Procese QED în câmpuri laser intense
tema 12

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”Verbum sapienti sat est”

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Capitol 1

Câmpuri clasice

1.1 De la mecanica clasică la câmpuri clasice şi cuantice

- Trecerea de la mecanica clasică la mecanica cuantică (cuantificarea I) se face prin

  înlocuirea *mărimi fizice* $\longrightarrow$ *operatori*

  Coordonata şi impulsul devin operatori $\quad \hat{x}=x \quad ; \quad \hat{p}=-i\hbar\frac{\partial}{\partial x}$

  Operatorii $\hat{x}$ şi $\hat{p}$ satisfac *relaţia de comutare* $\quad [\hat{x},\hat{p}]\equiv\hat{x}\hat{p}-\hat{p}\hat{x}=i\,\hbar \qquad (1.1)$

  Exemplu: Operatorul Hamilton pentru oscilatorul armonic este:
  $$\hat{H}=\frac{1}{2m}\hat{p}^2+\frac{1}{2}k\,\hat{x}^2 \qquad (1.2)$$

- In mecanica clasică, descriem *poziţia* prin coordonatele $\vec{x}$. Printr-o transformare de coordonate (ex. rotaţia unei bare) are loc transformarea reciprocă a coordonatelor ($x \leftrightarrow y$).

  La fel, câmpurile $\vec{E}$ şi $\vec{B}$ de exemplu, se pot transforma reciproc ($\vec{B} \leftrightarrow \vec{E}$).

  Ecuaţiile de mişcare din mecanică ne dădeau variaţia coordonatei x în funcţie de timp t.

  Aceasta arată că putem trata asemănător componentele câmpurilor cu cele de coordonate.

  In teoria (clasică a) câmpului vom descrie mărimea urmărită *câmpul* $\varphi(x,t)$ în mod similar în funcţie de variabila independentă, care de data asta este 4-coordonata x^μ. Atunci,

- Trecerea de la mecanica clasică la câmpuri clasice se face prin înlocuirea:

  Coordonate $x_i \rightarrow$ componente de câmp φ_i
  $$x_i \longrightarrow \varphi_i(\boldsymbol{x},t) \qquad (1.3)$$

  Viteze $\dot{x}_i \rightarrow$ derivate de câmp $\partial_\mu\varphi_i$
  $$\frac{dx_i}{dt} \longrightarrow \frac{\partial\varphi_i(\boldsymbol{x},t)}{\partial x^\mu} \qquad (1.4)$$

- Trecerea de la câmpuri clasice la câmpuri cuantice (cuantificarea a II-a) se face prin

  înlocuirea: *câmpuri clasice* $\rightarrow$ *operatori de câmp.*

  Câmpul φ devine operator: $\quad \varphi(x,t) \rightarrow \hat{\varphi}(x,t)$

1.2 De la mecanica clasică la mecanica cuantică

- Trecerea de la mecanica clasică la mecanica cuantică (cuantificarea I) se face prin înlocuirea mărimilor fizice cu operatori:

$$x \longrightarrow \hat{x} \qquad p \longrightarrow \hat{p} = -i\hbar\frac{\partial}{\partial x} \qquad H(x,p) \longrightarrow \hat{H}(\hat{x},\hat{p})$$
$$E \longrightarrow \hat{E} = i\hbar\frac{\partial}{\partial t} \qquad \left\{x,p\right\} \longrightarrow \frac{1}{i\hbar}\left[\hat{x},\hat{p}\right] \tag{1.5}$$

	Ecuaţia Nerelativistă (Schrödinger)	Ecuaţia Relativistă (Klein-Gordon)	
Relaţia energie-impuls	1-dimensional $E=\frac{p^2}{2m}$	1-dimensional $\frac{E^2}{c^2}=p^2+m^2c^2$	covariant Lorentz $p^2=m^2c^2$
Trecerea la operatori diferenţiali:	$E\to i\hbar\frac{\partial}{\partial t}$; $p\to -i\hbar\frac{\partial}{\partial x}\equiv -i\hbar\boldsymbol{\nabla}$		$p^\mu \to i\hbar\frac{\partial}{\partial x_\mu}\equiv i\hbar\partial^\mu$
Ec. de undă pt. particula liberă	$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}=0$	$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}+\frac{m^2c^2}{\hbar^2}\psi=0$	$\left(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2}\right)\psi=0$
Ec. complex conjugată	$-i\hbar\frac{\partial\psi^*}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}=0$	$\frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2}-\frac{\partial^2\psi^*}{\partial x^2}+\frac{m^2c^2}{\hbar^2}\psi^*=0$	$\left(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2}\right)\psi^*=0$
Inmulţim la stânga ec. de bază cu $-i\psi^*$ iar ec. c.c. la dreapta cu $i\psi$ apoi adunarea lor	$\underbrace{\left(\psi^*\frac{\partial\psi}{\partial t}+\frac{\partial\psi^*}{\partial t}\psi\right)}_{\frac{\partial}{\partial t}(\psi^*\psi)=\frac{\partial\rho}{\partial t}}+$ $\underbrace{-\frac{i\hbar}{2m}\left(\psi^*\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi^*}{\partial x^2}\psi\right)}_{-\frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\psi^*\frac{\partial\psi}{\partial x}-\frac{\partial\psi^*}{\partial x}\psi\right)=\frac{\partial j}{\partial x}}=0$	$\underbrace{\frac{i}{c^2}\left(\psi^*\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi^*}{\partial t^2}\psi\right)}_{\frac{i}{c^2}\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi\right)=\frac{i}{c}\frac{\partial\rho}{\partial(ct)}}+$ $\underbrace{-i\left(\psi^*\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi^*}{\partial x^2}\psi\right)}_{-i\frac{\partial}{\partial x}\left(\psi^*\frac{\partial\psi}{\partial x}-\frac{\partial\psi^*}{\partial x}\psi\right)=+\frac{\partial j}{\partial x}}=0$	$\begin{cases}x^\mu\equiv(ct,\vec{x})\ ;\ x_\mu\equiv(ct,-\vec{x})\\ \partial^\mu\equiv\left(\frac{\partial}{\partial(ct)},-\frac{\partial}{\partial x_\mu}\right)\ ;\\ \partial_\mu\equiv\left(\frac{\partial}{\partial(ct)},\frac{\partial}{\partial x_\mu}\right)\end{cases}$ $\frac{\partial}{\partial x^\mu}\underbrace{\left[i\left(\psi^*\frac{\partial\psi}{\partial x_\mu}-\frac{\partial\psi^*}{\partial x_\mu}\psi\right)\right]}_{j^\mu}=0$
Ecuaţia de continuitate	$\frac{\partial\rho}{\partial t}+\frac{\partial j}{\partial x}=0$	$\frac{1}{c}\frac{\partial\rho}{\partial t}+\frac{\partial j}{\partial x}=0$	$\partial_\mu j^\mu=0$
Unde Densitatea ρ şi Curentul de probab. j	$\begin{cases}\rho=\psi^*\psi=\|\psi\|^2\\ j=-\frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x}-\frac{\partial\psi^*}{\partial x}\psi\right)\end{cases}$	$\begin{cases}\rho=\frac{i}{c}\left(\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi\right)\\ j=-i\left(\psi^*\frac{\partial\psi}{\partial x}-\frac{\partial\psi^*}{\partial x}\psi\right)\end{cases}$	$j^0\equiv c\rho\quad ;\quad j^{1,2,3}\equiv\vec{j}$ $j^\mu=i\left(\psi^*\partial^\mu\psi-\partial^\mu\psi^*\psi\right)$
Folosind soluţia de undă plană a ecuaţiei de bază, cu înlocuirile: $\begin{cases}p\cdot x=p_\mu x^\mu=Et-\vec{p}\cdot\vec{x}\\ k\cdot x=k_\mu x^\mu=\omega t-\vec{k}\cdot\vec{x}=\frac{E}{\hbar}t-\frac{\vec{p}}{\hbar}\cdot\vec{x}\end{cases}$	$\longrightarrow$	$\begin{aligned}\psi&=N\,e^{-i(E\cdot t-\vec{p}\cdot\vec{x})/\hbar}\\ &\equiv N\,e^{-i(\omega\cdot t-\vec{k}\cdot\vec{x})}\\ \psi^*&=N^*\,e^{i(E\cdot t-\vec{p}\cdot\vec{x})/\hbar}\\ &\equiv N^*\,e^{i(\omega\cdot t-\vec{k}\cdot\vec{x})}\end{aligned}$	$\begin{aligned}\psi&=N\,e^{-i\,p\cdot x/\hbar}\\ &\equiv N\,e^{-i\,k\cdot x}\\ \psi^*&=N^*\,e^{i\,p\cdot x/\hbar}\\ &\equiv N^*\,e^{i\,k\cdot x}\end{aligned}$
Densitatea şi curentul de probabilitate pt. unda plană	$\begin{cases}\rho=\|N\|^2\\ \vec{j}=\frac{\vec{p}}{m}\|N\|^2\end{cases}$	$\begin{cases}c\rho=i\left(-2\,i\,\frac{E}{\hbar}\right)\|N\|^2=2\frac{E}{\hbar}\|N\|^2\\ \vec{j}=-i\left(2\,i\,\frac{\vec{p}}{\hbar}\right)\|N\|^2=2\frac{\vec{p}}{\hbar}\|N\|^2\end{cases}$	$j^\mu=2\,\frac{p^\mu}{\hbar}\|N\|^2$

<table>
<tr><td></td><td colspan="3">Ecuaţia Relativistă - Dirac</td></tr>
<tr><td>Relaţia energie-impuls</td><td>$\frac{E}{c}=\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta mc$</td><td>In reprezentarea Dirac-Pauli
$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\ \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$</td><td>In reprezentarea Weyl
$\boldsymbol{\alpha}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix},\ \beta=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$</td></tr>
<tr><td colspan="4">Trecerea la operatori diferenţiali: $E\to i\hbar\frac{\partial}{\partial t}$; $\boldsymbol{p}\to -i\hbar\boldsymbol{\nabla}$
$\Longrightarrow\quad i\hbar\frac{\partial\psi}{\partial ct}+i\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi=\beta\, mc\,\psi$ înmulţim la stânga cu β şi notăm: $\gamma^0=\beta,\quad \gamma^k=\beta\alpha^k$</td></tr>
<tr><td>Ecuaţia Dirac</td><td colspan="2">3-dimensional
$i\hbar\gamma^0\frac{\partial\psi}{\partial ct}+i\hbar\gamma^k\frac{\partial\psi}{\partial x^k}-mc\,\psi=0$</td><td>covariant Lorentz
$(i\hbar\gamma^\mu\partial_\mu-mc)\,\psi=0$</td></tr>
<tr><td>Ecuaţia Dirac conjugată hermitic</td><td colspan="2">$-i\hbar\frac{\partial\psi^\dagger}{\partial ct}\gamma^0-i\hbar\frac{\partial\psi^\dagger}{\partial x^k}\left(-\gamma^k\right)-mc\,\psi^\dagger=0$
$\psi^\dagger$ - conj.hermitic (transp.& c.c.) - linie</td><td>pentru a reface forma covariantă, înlăturăm minusul de la γ^k. De aceea, deoarece $\gamma^0\gamma^k=-\gamma^k\gamma^0$, înmulţim ecuaţia la dreapta cu γ^0.
notăm: $\bar{\psi}\equiv\psi^\dagger\gamma^0$ (spinor adjunct) - linie</td></tr>
<tr><td>Ec. Dirac adjunctă</td><td colspan="2">$i\hbar\frac{\partial\bar{\psi}}{\partial ct}\gamma^0+i\hbar\frac{\partial\bar{\psi}}{\partial x^k}\gamma^k+mc\,\bar{\psi}=0$</td><td>$i\hbar\partial_\mu\bar{\psi}\,\gamma^\mu+mc\,\bar{\psi}=0$</td></tr>
<tr><td>Inmulţim ecuaţia de bază cu $\bar{\psi}$ la stînga</td><td colspan="2">$i\hbar\bar{\psi}\gamma^0\frac{\partial\psi}{\partial ct}+i\hbar\bar{\psi}\gamma^k\frac{\partial\psi}{\partial x^k}-mc\,\bar{\psi}\psi\ +$</td><td rowspan="3">$\bar{\psi}\gamma^\mu\partial_\mu\psi+\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi=$
$=\partial_\mu\underbrace{\left(\bar{\psi}\gamma^\mu\psi\right)}_{j^\mu}=0$</td></tr>
<tr><td>apoi ecuaţia adjunctă cu ψ la dreapta</td><td colspan="2">$i\hbar\frac{\partial\bar{\psi}}{\partial ct}\gamma^0\psi+i\hbar\frac{\partial\bar{\psi}}{\partial x^k}\gamma^k\psi+mc\,\bar{\psi}\psi=0$</td></tr>
<tr><td>şi le adunăm</td><td colspan="2">$\frac{\partial}{\partial ct}\Big(\underbrace{\bar{\psi}\gamma^0\psi}_{\rho}\Big)+\frac{\partial}{\partial x^k}\Big(\underbrace{\bar{\psi}\gamma^k\psi}_{j^k}\Big)=0$</td></tr>
<tr><td colspan="3">$\Longrightarrow$ Ecuaţia de continuitate $\quad\frac{1}{c}\frac{\partial\rho}{\partial t}+\sum_{k=1}^{3}\frac{\partial j^k}{\partial x^k}$</td><td>$\partial_\mu j^\mu=0$</td></tr>
<tr><td>Densitatea şi curentul de probab.</td><td colspan="2">$\begin{cases}\rho\equiv j^0=\bar{\psi}\gamma^0\psi=\psi^\dagger\psi=\sum_{i=1}^{3}|\psi_i|^2>0\\ \vec{j}\equiv j^k=\bar{\psi}\gamma^k\psi=\psi^\dagger\gamma^0\gamma^k\psi\end{cases}$</td><td>$j^\mu=\bar{\psi}\gamma^\mu\psi$
dacă j^μ e 4-curentul de ELECTRONI, se înmulţeşte cu sarcina $-e$: $j^\mu=-e\,\bar{\psi}\gamma^\mu\psi$</td></tr>
</table>

1.3 De la mecanica clasică la câmpuri clasice

- Trecerea de la mecanica clasică la câmpuri clasice, se face prin înlocuirea variabilelor de coordonată şi derivata coordonatei (viteză sau impuls) cu cele de câmp şi derivata covariantă a câmpului:

$$\begin{array}{lll} x_i \longrightarrow \varphi_\mu & p \longrightarrow \pi & L(x,\dot{x}) \longrightarrow \mathcal{L}(\varphi,\dot{\varphi}) \\ \dot{x}_i \longrightarrow \partial_\mu \varphi & & \end{array} \tag{1.6}$$

1.3.1 Sistem discret n oscilatori legaţi - cu număr finit de grade de libertate

- Lagrangian

$$L=T-V=\frac{1}{2}m\left(\dot{y}_1^2+\dot{y}_2^2+\cdots+\dot{y}_n^2\right)- -\frac{k_t}{2}\left[(y_1-0)^2+(y_2-y_1)^2+\cdots+(0-y_n)^2\right] \tag{1.7}$$

- Ecuaţiile Euler-Lagrange $\dfrac{\partial L}{\partial y_i}-\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{y}_i}\right)=0$

$$\begin{cases} -k_t\left(2y_1-y_2\right)=m\ddot{y}_1 \\ -k_t\left(-y_1+2y_2-y_3\right)=m\ddot{y}_2 \\ -k_t\left(-y_2+2y_3-y_4\right)=m\ddot{y}_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ -k_t\left(-y_{n-2}+2y_{n-1}-y_n\right)=m\ddot{y}_{n-1} \\ -k_t\left(-y_{n-1}+2y_n\right)=m\ddot{y}_n \end{cases}$$

sau

$$\omega_0^2\left(y_{i-1}-2y_i+y_{i+1}\right)=\ddot{y}_i \quad (i=1,\dots,n) \tag{1.8}$$

unde am notat $\omega_0^2=\dfrac{\tau}{ma}=\dfrac{k_t}{m}$

- Soluţii

$$y_i(x_i,t)=\underbrace{A\sin(\kappa\, x_i)}\cos(\omega t-\phi) \qquad (i=1,\dots,n) \tag{1.9}$$

- Cuantificarea oscilaţiilor staţionare

Cu condiţia la margine $y_{n+1}=0$ $\left(x_{n+1}=(n+1)a\right)$, din (1.9) avem:

$$\sin(\kappa\, x_{n+1})=\sin\underbrace{\left[\kappa(n+1)\,a\right]}_{=j\,\pi}=0$$

- Valorile cuantificate (permise) pentru vectorul de undă κ_j vor fi:

$$\kappa_j=\frac{j\pi}{a(n+1)} \qquad (j=1,2,\dots,n) \tag{1.10}$$

Folosind cuantificarea vectorului de undă κ_j (1.10) şi legătura (**??**) cu frecvenţele ω_j, avem şi cuantificarea frecvenţelor pentru fiecare grad de libertate j al ansamblului oscilant:

$$\omega_j=2\omega_0\sin\left(\kappa_j\frac{a}{2}\right)=2\,\omega_0\sin\left(\frac{j\pi}{2(n+1)}\right) \quad (j=1,2,\dots,n) \tag{1.11}$$

- Hamiltonian | (1.12)

1.3.2 Sistem finit cu $n \to \infty$ mase oscilante - coardă elastică

- Lagrangian

$$L = T - V = \frac{m}{2}\sum_{i=1}^{n} \dot{y}_i^2 - \frac{\tau}{2a}\sum_{i=0}^{n}(y_{i+1}-y_i)^2 \tag{1.13}$$

- Ecuaţiile Euler-Lagrange

$$\frac{\partial L}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) = 0$$

$$\frac{\tau}{a}\Big[\Big(y(x+a)-y(x)\Big) - \Big(y(x)-y(x-a)\Big)\Big] = \overbrace{\rho\, a}^{m}\, \ddot{y}(x) \tag{1.14}$$

$$\left.\begin{array}{c}\text{la limita}\\ a \equiv \Delta x \to 0\end{array}\right\} \quad \tau\,\frac{\partial^2 y}{\partial x^2} - \rho\,\frac{\partial^2 y}{\partial t^2} = 0 \tag{1.15}$$

Ec. propagare unde cu $v^2 = \tau/\rho$

- Soluţii

$$y(x,t) = \underbrace{A\sin(\kappa\, x)}_{\text{comp. spaţială}}\ \cos(\omega t - \phi) \quad \text{unde} \quad \kappa = \frac{2\pi}{\lambda} \tag{1.16}$$

- Cuantificarea oscilaţiilor staţionare

Din condiţia la margine $x_n = \ell$ $y(\ell) = 0$ $\big(\ell = na\big)$, componenta spaţială din (1.16) dă:

$$\sin(\kappa\, x_n) = \sin\underbrace{\Big[\kappa\, n\, a\Big]}_{=j\,\pi} = 0$$

$na = \ell$ (lungimea corzii)

- Valorile cuantificate (permise) pentru vectorul de undă κ_j vor fi:

$$\kappa_j = \frac{j\pi}{n\,a} \qquad (j = 1, 2, \dots) \tag{1.17}$$

Prin cuantificarea (1.17) a vectorului de undă κ_j şi legătura (**??**) cu frecvenţele ω_j, avem şi cuantificarea frecvenţelor pentru fiecare grad de libertate j din cele n ale ansamblului oscilant:

cu $\quad \omega_0^2 = \dfrac{\tau}{m\,a} = \dfrac{k_t}{m}$

$$\omega_j = 2\omega_0 \sin\left(\kappa_j \frac{a}{2}\right) = 2\,\omega_0 \sin\left(\frac{j\pi}{2\,n}\right) \qquad (j = 1, 2, \dots) \tag{1.18}$$

- Lagrangian pentru o distribuţie continuă de mase (coardă elastică). A se identifica termenii din (1.13).

$$L = T - V = \int_0^{\ell} \mathcal{L}(x)\, dx \qquad \text{unde}$$

$$\mathcal{L}(x) = \underbrace{\frac{\rho}{2}\left(\frac{\partial y(x,t)}{\partial t}\right)^2}_{T} - \underbrace{\frac{\rho\, v^2}{2}\left(\frac{\partial y(x,t)}{\partial x}\right)^2}_{V} \tag{1.19}$$

- Hamiltonian pentru o distribuţie continuă de mase (coardă elastică)

$$\mathcal{H} = \underbrace{\frac{\rho}{2}\left(\frac{\partial y(x,t)}{\partial t}\right)^2}_{T} + \underbrace{\frac{\rho\, v^2}{2}\left(\frac{\partial y(x,t)}{\partial x}\right)^2}_{V} \tag{1.20}$$

1.3.3 Câmp scalar ϕ - sistem continuu cu număr infinit de grade de libertate

- Trecerea de la mecanica clasică la câmpuri clasice se face conform

$$x_i \longrightarrow \phi_\mu \qquad L(x,\dot{x}) \longrightarrow \mathcal{L}(\phi,\partial_\mu\phi)$$
$$\dot{x}_i \longrightarrow \partial_\mu\phi \qquad p=\frac{\partial L}{\partial \dot{x}} \longrightarrow \pi=\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{1.21}$$

- Densitatea de Lagrangian şi de impuls de câmp similar cu $p=\dfrac{\partial L}{\partial \dot{q}}$

$$L(\phi,\partial_t\phi,\partial_x\phi)=\int dx\,\mathcal{L}(\phi,\partial_t\phi,\partial_x\phi) \tag{1.22}$$

$$\mathcal{L}=\overbrace{\frac{1}{2}\frac{1}{c^2}\Big(\partial_t\phi\Big)^2}^{T}-\overbrace{\frac{1}{2}\Big(\partial_x\phi\Big)^2}^{V}$$
$$\pi(x,t)=\frac{\partial\mathcal{L}}{\partial\left(\partial_t\phi\right)}=\frac{1}{c^2}\partial_t\phi \tag{1.23}$$

- Ecuaţiile Euler-Lagrange $\dfrac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)=0$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \underbrace{\left(\frac{1}{c^2}\partial_t^2-\partial_x^2\right)\phi=0}_{\text{Ec. propagare unde cu } c^2} \tag{1.24}$$

- Hamiltonianul din mecanică $H(q,p)=p\,\dot{q}-L$ devine Densitatea de Hamiltonian $\mathcal{H}(\phi,\pi)=\pi\,(\partial_t\phi)-\mathcal{L}$

$$\mathcal{H}(\phi,\pi)=\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\partial_t\phi-\frac{1}{2}\frac{1}{c^2}\left(\partial_t\phi\right)^2+\frac{1}{2}\left(\partial_x\phi\right)^2$$
$$=\overbrace{\frac{1}{2}\frac{1}{c^2}\Big(\partial_t\phi\Big)^2}^{T}+\overbrace{\frac{1}{2}\Big(\partial_x\phi\Big)^2}^{V} \tag{1.25}$$

1.3.4 Câmp scalar real masiv φ (Klein-Gordon)

Descrie un sistem infinit de oscilatori armonici cuplaţi.

- Densitatea de Lagrangian (cu $\hbar = c = 1$)

$$\mathcal{L}=\frac{1}{2}\left[(\partial_\mu\varphi)^2-m^2\varphi^2\right]=\frac{1}{2}\left[(\partial_t\varphi)^2-(\nabla\varphi)^2-m^2\varphi^2\right] \tag{1.26}$$

unde $(\partial_\mu\varphi)^2=\partial_\mu\varphi\,\partial^\mu\varphi$ iar $\pi=\frac{\partial\mathcal{L}}{\partial_t\varphi}=\partial^t\varphi=\partial_t\varphi$

- Ecuaţiile Euler-Lagrange $\frac{\partial\mathcal{L}}{\partial\varphi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)=0$

$$\left(\partial_\mu\partial^\mu + m^2\right)\varphi=\left(\partial_t^2 - \nabla^2 + m^2\right)\varphi = 0 \tag{1.27}$$

- Densitatea de Hamiltonian (analog $H = p\dot{q} - L$)

$$\begin{aligned}\mathcal{H} &= \pi\partial_t\varphi-\mathcal{L}=(\partial_t\varphi)^2-\frac{1}{2}\left[(\partial_t\varphi)^2-(\nabla\varphi)^2-m^2\varphi^2\right]\\ &=\frac{1}{2}\left[(\partial_t\varphi)^2+(\nabla\varphi)^2+m^2\varphi^2\right]\end{aligned} \tag{1.28}$$

1.3.5 Câmp scalar complex masiv φ şi φ^* (Klein-Gordon)

• Densitatea de Lagrangian (cu $\hbar = c = 1$)	$$\mathcal{L}=\partial_\mu\varphi^*\partial^\mu\varphi-m^2\varphi^*\varphi \equiv \dot{\varphi}^*\dot{\varphi} -\nabla\varphi^*\cdot\nabla\varphi-m^2\varphi^*\varphi \quad (1.29)$$
• Ecuaţiile Euler-Lagrange $\frac{\partial\mathcal{L}}{\partial\varphi_i}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)=0$	$$\left(\partial_\mu\partial^\mu + m^2\right)\varphi^* \equiv \left(\partial_t^2 - \nabla^2 + m^2\right)\varphi^* = 0 \quad (1.30)$$ $$\left(\partial_\mu\partial^\mu + m^2\right)\varphi \equiv \left(\partial_t^2 - \nabla^2 + m^2\right)\varphi = 0 \quad (1.31)$$
• Soluţiile de *câmp liber scalar real* $\varphi(\vec{x},t)$ şi complex $\varphi^*(\vec{x},t)$ sunt unde plane (vezi Tabel pag. **??**):	$$\begin{cases}\varphi(\vec{x},t) = \alpha(\vec{k})\, e^{-i(\omega t-\vec{k}\cdot\vec{x})} \\ \varphi^*(\vec{x},t) = \alpha^*(\vec{k})\, e^{i(\omega t-\vec{k}\cdot\vec{x})}\end{cases} \quad (1.32)$$
• Soluţia generală de câmp scalar este o superpoziţie (dezvoltare Fourier (vezi (**??**)) de unde plane φ şi φ^*. $k\cdot x=\omega t-\vec{k}\cdot\vec{x}$, iar $d^4k = dk^0\, d^3\vec{k} = dE\, d^3\vec{p}=d^4p$ şi integrăm după energie.	$$\varphi(\vec{x},t)=\int d^4k\left(\alpha_{\vec{k}}\, e^{-ik\cdot x}+\alpha^*_{\vec{k}}\, e^{ik\cdot x}\right) \quad (1.33)$$ efectuând integrarea după dE cu condiţia de conservare a energiei: $\delta(p^2-m^2)=\delta(E^2-\lvert\vec{p}\rvert^2-m^2)$, $$\varphi(x)=\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_{\vec{k}}}}\left(\alpha_{\vec{k}}\, e^{-ik\cdot x}+\alpha^*_{\vec{k}}\, e^{ik\cdot x}\right) \quad (1.34)$$

1.3.6 Câmp Schrödinger real şi complex

• Densitatea de Lagrangian	$$\mathcal{L} = i\hbar\varphi^*\dot{\varphi} - \frac{\hbar^2}{2m}\nabla\varphi^*\cdot\nabla\varphi - V\varphi^*\varphi \quad (1.35)$$
• Ecuaţiile Euler-Lagrange $$\frac{\partial\mathcal{L}}{\partial\varphi_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right) = 0$$	$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\varphi + V\,\varphi \quad (1.36)$$ $$-i\hbar\frac{\partial\varphi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\varphi^* + V\varphi^* \quad (1.37)$$
• Soluţiile de *câmp liber Schrödinger real* $\varphi(\vec{x},t)$ şi complex $\varphi^*(\vec{x},t)$ sunt unde plane:	$$\begin{cases} \varphi(\vec{x},t) = \alpha\, e^{-i(\omega t - \vec{k}\cdot\vec{x})} \\ \varphi^*(\vec{x},t) = \alpha^*\, e^{i(\omega t - \vec{k}\cdot\vec{x})} \end{cases} \quad (1.38)$$
• Soluţia generală de câmp Schrödinger este o superpoziţie (dezvoltare Fourier (vezi (**??**)) de unde plane φ şi φ^*. $k\cdot x = \omega t - \vec{k}\cdot\vec{x}$, iar $d^4k = dk^0\, d^3\vec{k} = dE\, d^3\vec{p} = d^4p$ şi integrăm după energie.	$$\varphi(\vec{x},t) = \int d^4k\, \left(\alpha_{\vec{k}}\, e^{-ik\cdot x} + \alpha^*_{\vec{k}}\, e^{ik\cdot x}\right) \quad (1.39)$$ efectuând integrarea după dE cu condiţia de conservare a energiei: $\delta(p^2 - m^2) = \delta(E^2 - \|\vec{p}\|^2 - m^2)$, $$\varphi(x) = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3\, 2\omega_{\vec{k}}}}\left(\alpha_{\vec{k}}\, e^{-ik\cdot x} + \alpha^*_{\vec{k}}\, e^{ik\cdot x}\right) \quad (1.40)$$

1.3.7 Câmp Dirac

• Densitatea de Lagrangian	$$\mathcal{L} = i\,\hbar\,\bar{\varphi}\,\gamma^{\mu}\partial_{\mu}\varphi - m\,\bar{\varphi}\,\varphi \qquad (1.41)$$ unde $\bar{\varphi} = \varphi^{\dagger}\gamma^{0}$
• Ecuaţiile Euler-Lagrange $\dfrac{\partial\mathcal{L}}{\partial\varphi_i} - \partial_{\mu}\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_{\mu}\varphi_i)}\right) = 0$	$$i\,\hbar\,\gamma^{\mu}\partial_{\mu}\varphi - mc\,\varphi = 0 \qquad (1.42)$$ $$i\,\hbar\,\partial_{\mu}\bar{\varphi}\,\gamma^{\mu} + m\,c\,\bar{\varphi} = 0 \qquad (1.43)$$
• Soluţiile de *câmp liber Dirac real* $\varphi(\vec{x},t)$ şi adjunct $\bar{\varphi}(\vec{x},t)$ sunt unde plane:	$$\begin{cases} \varphi(\vec{x},t) \;=\; u(\vec{p})\,e^{-i\,p\cdot x} \\ \bar{\varphi}(\vec{x},t) \;=\; v(\vec{p})\,e^{i\,p\cdot x} \end{cases} \qquad (1.44)$$
• Spinorii	$$u^{(1,2)} = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \end{pmatrix} \qquad v^{(1,2)} = N\begin{pmatrix} \dfrac{-\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\lvert E\rvert+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{-\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\lvert E\rvert+m} \\ 0 \\ 1 \end{pmatrix} \qquad (1.45)$$

1.3.8 Câmp Maxwell

• Densitatea de Lagrangian	$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu_0 j^\mu A_\mu \quad (1.46)$$ $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad ; \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (1.47)$$
• Ecuaţiile Euler-Lagrange $$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0$$	$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad (1.48)$$

Capitol 2

Câmpuri cuantice

- Trecerea de la câmpuri clasice la câmpuri cuantice (cuantificarea a II-a) se face prin înlocuirea câmpurilor şi a derivatelor de câmp cu operatorii corespunzători.

$$\begin{array}{lll} \varphi \longrightarrow \hat{\varphi} & \pi \longrightarrow \hat{\pi} & \mathcal{H}(\varphi,\pi) \longrightarrow \hat{\mathcal{H}}(\hat{\varphi},\hat{\pi}) \\ \varphi^* \longrightarrow \hat{\varphi}^\dagger & \pi^* \longrightarrow \hat{\pi}^\dagger & \Big\{\varphi,\pi\Big\} \longrightarrow \dfrac{1}{i\hbar}\left[\hat{\varphi},\hat{\pi}\right] \end{array} \tag{2.1}$$

2.0.9 Oscilatorul armonic în mecanica cuantică şi teoria câmpurilor cuantice

- In cazul oscilatorului armonic, înlocuim operatorii $\hat{x}$ şi $\hat{p}$ prin $\hat{a}$ şi $\hat{a}^\dagger$ (complex-conjugat-ul lui $\hat{a}$) astfel ca relaţia de comutare (1.1) să devină $\left[\hat{a},\hat{a}^\dagger\right]=1$ (vezi (2.4)).

- Inlocuim în Hamiltonian (1.2) operatorii $\hat{x}$ şi $\hat{p}$ prin $\hat{a}$ şi $\hat{a}^\dagger$ complex conjugaţi:

$$\boxed{\hat{a}=\frac{1}{\sqrt{2\hbar\omega}}\left(\sqrt{k}\,\hat{x}+i\,\frac{\hat{p}}{\sqrt{m}}\right)} \quad ; \quad \boxed{\hat{a}^\dagger=\frac{1}{\sqrt{2\hbar\omega}}\left(\sqrt{k}\,\hat{x}-i\,\frac{\hat{p}}{\sqrt{m}}\right)} \tag{2.2}$$

Coeficienţii s-au ales astfel încât $[\hat{x},\hat{p}]=i\hbar$ trece în $[\hat{a},\hat{a}^\dagger]=1$. Se verifică imediat:

$$\hat{a}\hat{a}^\dagger=\frac{1}{2\hbar\omega}\left(k\hat{x}^2-i\sqrt{\frac{k}{m}}\hat{x}\hat{p}+i\sqrt{\frac{k}{m}}\hat{p}\hat{x}+\frac{1}{m}\hat{p}^2\right)=\frac{1}{\hbar\omega}\left(\frac{1}{2}k\hat{x}^2+\frac{1}{2m}\hat{p}^2-\frac{i}{2}\omega\,(i\,\hbar)\right)=\frac{1}{\hbar\omega}\left(\hat{H}+\frac{1}{2}\hbar\omega\right)$$

$$\hat{a}^\dagger\hat{a}=\frac{1}{2\hbar\omega}\left(k\hat{x}^2+i\sqrt{\frac{k}{m}}\hat{x}\hat{p}-i\sqrt{\frac{k}{m}}\hat{p}\hat{x}+\frac{1}{m}\hat{p}^2\right)=\frac{1}{\hbar\omega}\left(\frac{1}{2}k\hat{x}^2+\frac{1}{2m}\hat{p}^2+\frac{i}{2}\omega\,(i\,\hbar)\right)=\frac{1}{\hbar\omega}\left(\hat{H}-\frac{1}{2}\hbar\omega\right)$$

- Din fiecare expresie de mai sus putem exprima Hamiltonian-ul: $$\hat{H}=\hbar\omega\left(\hat{a}\hat{a}^\dagger-\frac{1}{2}\right) \quad \text{şi} \quad \hat{H}=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) \tag{2.3}$$

- Comutatorul: $$\left[\hat{a},\hat{a}^\dagger\right]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=\frac{1}{\hbar\omega}\left(\hat{H}+\frac{1}{2}\hbar\omega\right)-\frac{1}{\hbar\omega}\left(\hat{H}-\frac{1}{2}\hbar\omega\right)=\frac{1}{2}+\frac{1}{2}=1 \tag{2.4}$$

- Comutatorii: $$\left[\hat{a},\hat{H}\right]=\hbar\omega\hat{a} \quad ; \quad \left[\hat{a}^\dagger,\hat{H}\right]=-\hbar\omega\hat{a}^\dagger \tag{2.5}$$

- Din (2.2), cu $k=m\omega^2$, exprimăm vechii operatori $\hat{x}$ şi $\hat{p}$ prin cei noi $\hat{a}$ şi $\hat{a}^\dagger$

$$\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right) \quad ; \quad \hat{p}=-i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}-\hat{a}^\dagger\right) \tag{2.6}$$

- Prin adunarea expresiilor (2.3), rezultă Hamiltonian-ul oscilatorului armonic: $$\hat{H}=\frac{1}{2}\hbar\omega\left(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right) \tag{2.7}$$

2.1 Cuantificarea câmpurilor (cuantificarea a II-a)

- Cuantificarea *câmpurilor clasice* pentru a obţine *câmpuri cuantice* se face asemănător cum s-a trecut de la *mecanica clasică* la *mecanica cuantică* prin trecerea de la mărimile fizice la operatori. Aici trecerea se face de la câmpuri (funcţii de valori reale) la operatori hermitici $\psi(x)\to\hat{\psi}(x)$ şi $\pi(x)\to\hat{\pi}(x)$.

2.1.1 Ecuaţiile de câmp cuantic Schrödinger real $\hat{\psi}$

- Ecuaţiile de mişcare pentru operatori $\hat{\psi}_i(\vec{x},t)$; $\hat{\psi}_i^\dagger(\vec{x},t)$ (ecuaţiile Heisenberg) sunt

$$i\,\hbar\frac{d\hat{\psi}_i(x)}{dt}=\left[\hat{\psi}_i(x),\hat{H}_{Sch}\right] \qquad ; \qquad i\,\hbar\frac{d\hat{\psi}_i^\dagger(x)}{dt}=\left[\hat{\psi}_i^\dagger(x),\hat{H}_{Sch}\right] \tag{2.8}$$

- Hamiltonian-ul de câmp Schrödinger exprimat prin operatorii $\hat{\psi}$ şi $\hat{\psi}^\dagger$ (ψ^* e înlocuit cu operatorul hermitic adjunct $\hat{\psi}^\dagger$):

$$\hat{H}_{Sch}=\int\left(\frac{\hbar^2}{2m}\hat{\nabla}\hat{\psi}^\dagger\cdot\hat{\nabla}\hat{\psi}+V\hat{\psi}^\dagger\hat{\psi}\right)d^3\vec{x} \tag{2.9}$$

- *Relaţiile de comutare* pentru $\hat{\psi}$ şi $\hat{\psi}^\dagger$ la acelaşi t_0, se obţin prin trecerea $\hat{x}\to\hat{\psi}$ şi $\hat{p}\to\hat{\pi}=i\hbar\hat{\psi}^\dagger$ a relaţiilor $[\hat{x}_i,\hat{x}_j]=[\hat{p}_i,\hat{p}_j]=0$ şi $[\hat{x}_i,\hat{p}_j]=i\hbar\delta_{ij}$

$$\begin{aligned}&\left[\hat{\psi}(\vec{x},t_0),\hat{\psi}(\vec{x}',t_0)\right]=\left[\hat{\psi}^\dagger(\vec{x},t_0),\hat{\psi}^\dagger(\vec{x}',t_0)\right]=0\\ &\left[\hat{\psi}(\vec{x},t_0),\hat{\psi}^\dagger(\vec{x}',t_0)\right]=i\hbar\delta^3(\vec{x}-\vec{x}')\end{aligned} \tag{2.10}$$

- Pentru câmp Schrödinger real evaluăm comutat. (notăm $\hat{\psi}'\equiv\hat{\psi}(\vec{x}')$ ş.a.)

$$\left[\hat{\psi},\hat{H}_{Sch}\right]=\left[\hat{\psi},\int\frac{\hbar^2}{2m}\hat{\nabla}'\hat{\psi}'^\dagger\cdot\hat{\nabla}'\psi' d^3\vec{x}'\right]+\left[\hat{\psi},\int V'\hat{\psi}'^\dagger\hat{\psi}'d^3\vec{x}'\right]$$

- In 1-ul termen facem integrarea prin părţi

$$\int\underbrace{\hat{\nabla}'\hat{\psi}'^\dagger}_{dv}\cdot\underbrace{\hat{\nabla}'\hat{\psi}'}_{u}\,d^3\vec{x}'=-\int\hat{\psi}'^\dagger\hat{\nabla}'^2\hat{\psi}'d^3\vec{x}'$$

termenul
$uv \to 0$
$\hat{\psi} \to 0$
pt. $x \to \infty$

$$\left[\hat{\psi},\int\hat{\nabla}'\hat{\psi}'^\dagger\cdot\hat{\nabla}'\hat{\psi}'d^3x'\right]=-\left[\hat{\psi},\int\hat{\psi}'^\dagger\hat{\nabla}'^2\hat{\psi}'d^3x'\right]=-\int\left[\hat{\psi},\hat{\psi}'^\dagger\right]\hat{\nabla}'^2\hat{\psi}'d^3x'=-\int\delta^3(\vec{x}-\vec{x}')\hat{\nabla}'^2\hat{\psi}'d^3x'=-\hat{\nabla}^2\hat{\psi}$$

- In al 2-lea termen, folosim relaţiile de comutare (2.10), acesta devine:

$$\int V'(\hat{\psi}\hat{\psi}'^\dagger\hat{\psi}'-\hat{\psi}'^\dagger\underbrace{\hat{\psi}'\hat{\psi}}_{=\hat{\psi}\hat{\psi}'})d^3\vec{x}'=\int V'\left(\hat{\psi}\hat{\psi}'^\dagger-\hat{\psi}'^\dagger\hat{\psi}\right)\hat{\psi}'d^3\vec{x}=\int V'\hat{\psi}'\delta^3(\vec{x}-\vec{x}')d^3x'=V\hat{\psi}$$

- Ecuaţia de mişcare (2.8) pentru operatorul $\hat{\psi}(x)$ de câmp Schrödinger este:

$$\boxed{i\,\hbar\frac{d\hat{\psi}}{dt}=\left[\hat{\psi},\hat{H}_{Sch}\right]=-\frac{\hbar^2}{2m}\hat{\nabla}^2\hat{\psi}+V\hat{\psi}} \tag{2.11}$$

Am regăsit ecuaţia Schrödinger din mecanica cuantică pentru particulă.

2.1.2 Ecuaţiile de câmp cuantic scalar (Klein-Gordon) $\hat{\varphi}$

- Ecuaţia de mişcare pentru operatorul Heisenberg $\hat{\varphi}_i(\vec{x},t)$ este:

$$i\,\hbar\frac{\partial\hat{\varphi}_i(x)}{\partial t}=\Big[\hat{\varphi}_i(x),\hat{H}_{KG}\Big] \tag{2.12}$$

- Hamiltonian-ul de câmp Klein-Gordon exprimat prin $\hat{\varphi}$ şi $\hat{\pi}$:

$$\hat{H}_{KG}=\int\frac{1}{2}\left(\hat{\pi}^2(x)+\left(\hat{\nabla}\hat{\varphi}(x)\right)^2+m^2\hat{\varphi}^2(x)\right)d^3\vec{x} \tag{2.13}$$

deoarece $i\hbar\frac{d\hat{H}}{dt}=\Big[\hat{H},\hat{H}\Big]\equiv 0\Rightarrow \hat{H}_{KG}$ e independent de t

- *Relaţiile de comutare* pentru $\hat{\varphi}$ şi $\hat{\pi}$ la acelaşi t_0, se obţin prin trecerea $\hat{x}\to\hat{\varphi}$ şi $\hat{p}\to\hat{\pi}=\hat{\dot{\varphi}}$ a relaţiilor $[\hat{x}_i,\hat{x}_j]=[\hat{p}_i,\hat{p}_j]=0$ şi $[\hat{x}_i,\hat{p}_j]=i\hbar\delta_{ij}$

$$\begin{gathered}\Big[\hat{\varphi}(\vec{x},t_0),\hat{\varphi}'(\vec{x}',t_0)\Big]=\Big[\hat{\pi}(\vec{x},t_0),\hat{\pi}'(\vec{x}',t_0)\Big]=0\\ \Big[\hat{\varphi}(\vec{x},t_0),\hat{\pi}'(\vec{x}',t_0)\Big]=i\hbar\,\delta^3(\vec{x}-\vec{x}')\end{gathered} \tag{2.14}$$

- Pentru câmpul Klein-Gordon real evaluăm comutatorul

$$\Big[\hat{\varphi},\hat{H}_{KG}\Big]=\int\Big[\hat{\varphi}(\vec{x}),\hat{\mathcal{H}}_{KG}(\vec{x}')\Big]d^3\vec{x} \quad \text{adică conform (2.13) evaluăm} \quad \begin{cases}[\hat{\varphi},\hat{\pi'}^2]\\ [\hat{\varphi},(\hat{\nabla}'\hat{\varphi}')^2]\\ [\hat{\varphi},\hat{\varphi'}^2]\end{cases}$$

- Pentru comutatorul 1, $[\hat{\varphi},\hat{\pi}']=i\hbar\,\delta^3(\vec{x}'-\vec{x})$ care e o valoare complexă (comută), atunci

$$\begin{cases}\Big[\hat{\varphi},\hat{\pi'}^2\Big]=\hat{\varphi}\hat{\pi'}^2-\hat{\pi'}^2\hat{\varphi}=\underbrace{\hat{\varphi}\hat{\pi}'}_{i\hbar\delta^3+\hat{\pi}'\hat{\varphi}}\hat{\pi}'-\hat{\pi}'\underbrace{\hat{\pi}'\hat{\varphi}}_{\hat{\varphi}\hat{\pi}'-i\hbar\delta^3}=\\ =i\hbar\delta^3\hat{\pi}'\cancel{+}\hat{\pi}'\hat{\varphi}\,\hat{\pi}'-\cancel{\hat{\pi}'}\hat{\varphi}\,\hat{\pi}'+i\hbar\delta^3\hat{\pi}'=2i\hbar\delta^3\hat{\pi}'\end{cases} \tag{2.15}$$

- Pentru comutatorii 2 şi 3 avem $[\hat{\varphi}(\vec{x}),\hat{\varphi}'(\vec{x}')]=0\implies$

$$\begin{cases}[\hat{\varphi},\hat{\varphi'}^2]=0\\ [\hat{\varphi},\hat{\nabla}'\hat{\varphi}']=0\Rightarrow[\hat{\varphi},(\hat{\nabla}'\hat{\varphi}')^2]=0\end{cases}$$

- Atunci, comutatorul căutat este: $\Big[\hat{\varphi},\hat{H}_{KG}\Big]=\int i\hbar\,\delta^3(\vec{x}'-\vec{x})\,\hat{\pi}'(\vec{x}')=i\hbar\,\hat{\pi}(\vec{x})$

- iar ecuaţia de mişcare (2.12) pentru operatorul de câmp Klein-Gordon $\hat{\varphi}(x)$ este:

$$i\hbar\frac{\partial\hat{\varphi}}{\partial t}=\left[\hat{\varphi},\hat{H}_{KG}\right]=i\hbar\,\hat{\pi}(\vec{x})$$

sau

$$\boxed{\frac{\partial\hat{\varphi}}{\partial t}=\hat{\pi}(\vec{x})} \tag{2.16}$$

2.1.3 Ecuaţiile de câmp cuantic scalar (Klein-Gordon) $\hat{\pi}$

- Ecuaţia de mişcare pentru operatorul Heisenberg $\hat{\pi}_i(\vec{x},t)$ este:

$$i\hbar\frac{\partial\hat{\pi}_i(x)}{\partial t}=\left[\hat{\pi}_i(x),\hat{H}_{KG}\right] \tag{2.17}$$

- Hamiltonian-ul de câmp Klein-Gordon exprimat prin $\hat{\varphi}$ şi $\hat{\pi}$:

$$\hat{H}_{KG}=\int\frac{1}{2}\left(\hat{\pi}^2(x)+\left(\hat{\nabla}\hat{\varphi}(x)\right)^2+m^2\hat{\varphi}^2(x)\right)d^3\vec{x} \tag{2.18}$$

deoarece $i\hbar\frac{d\hat{H}}{dt}=\left[\hat{H},\hat{H}\right]\equiv 0 \Rightarrow \hat{H}_{KG}$ e independent de t

- Pentru impulsul $\hat{\pi}$ Klein-Gordon evaluăm comutatorul

$$\left[\hat{\pi},\hat{H}_{KG}\right]=\int\left[\hat{\pi}(\vec{x}),\hat{\mathcal{H}}_{KG}(\vec{x}')\right]d^3\vec{x} \qquad \text{folosind (2.14) şi (2.18)} \begin{cases}[\hat{\pi},\hat{\pi}'^2]\\ [\hat{\pi},(\hat{\nabla}'\hat{\varphi}')^2]\\ [\hat{\pi},\hat{\varphi}'^2]\end{cases}$$

- Pentru comutatorul 1 avem: $[\hat{\pi},\hat{\pi}']=0 \Rightarrow \left[\hat{\pi},\hat{\pi}'^2\right]=0$
- Pentru comutatorul 3 (folosind relaţia de comutare (2.14), avem: $[\hat{\pi},\hat{\varphi}']=-i\hbar\delta^3(\vec{x}'-\vec{x})$ apoi, la fel ca în (2.15) $\Longrightarrow \left[\hat{\pi},\hat{\varphi}'^2\right]=-2i\hbar\delta^3(\vec{x}'-\vec{x})\cdot\hat{\varphi}'$
- Pentru comutatorul 2:

$$[\hat{\pi},\hat{\nabla}'\hat{\varphi}']=\hat{\nabla}'[\hat{\pi},\hat{\varphi}']=-i\hbar\hat{\nabla}'\delta^3(\vec{x}'-\vec{x})$$

$$[\hat{\pi},(\hat{\nabla}'\hat{\varphi}')^2]=\hat{\pi}\left(\hat{\nabla}'\hat{\varphi}'\right)^2-\left(\hat{\nabla}'\hat{\varphi}'\right)^2\hat{\pi}=\overbrace{\hat{\pi}\hat{\nabla}'\hat{\varphi}'}^{-i\hbar\hat{\nabla}'\delta^3+\hat{\nabla}'\hat{\varphi}'\hat{\pi}}\,\hat{\nabla}'\hat{\varphi}'-\hat{\nabla}'\hat{\varphi}'\,\overbrace{\hat{\nabla}'\hat{\varphi}'\hat{\pi}}^{-i\hbar\hat{\nabla}'\delta^3+\hat{\pi}\hat{\nabla}'\hat{\varphi}'}=$$

$$=-i\hbar\hat{\nabla}'\delta^3\hat{\nabla}'\hat{\varphi}'+\cancel{\hat{\nabla}'\hat{\varphi}'\hat{\pi}\hat{\nabla}'\hat{\varphi}'}-i\hbar\hat{\nabla}'\hat{\varphi}'\hat{\nabla}'\delta^3-\cancel{\hat{\nabla}'\hat{\varphi}'\hat{\pi}\hat{\nabla}'\hat{\varphi}'}=-2i\hbar\delta^3\hat{\nabla}'^2\hat{\varphi}'$$

- Comutatorul căutat este:

$$\left[\hat{\pi},\hat{H}_{KG}\right]=\int\left[i\hbar\,\delta^3(\vec{x}'-\vec{x})\left(\hat{\nabla}'^2\hat{\varphi}'-m^2\hat{\varphi}'\right)\right]d^3\vec{x}'=i\hbar\hat{\nabla}^2\hat{\varphi}-i\hbar m^2\hat{\varphi}$$

- iar ecuaţia de mişcare (2.17) pentru operatorul impuls Klein-Gordon $\hat{\pi}(x)$ este:

$$i\hbar\frac{\partial\hat{\pi}}{\partial t}=\left[\hat{\pi},\hat{H}_{KG}\right]=i\hbar\left(\hat{\nabla}^2-m^2\right)\hat{\varphi}$$

sau

$$\boxed{\frac{\partial\hat{\pi}(x)}{\partial t}=\left(\hat{\nabla}^2-m^2\right)\hat{\varphi}(x)} \tag{2.19}$$

- Combinând ecuaţiile $\dfrac{\partial\hat{\varphi}}{\partial t}=\hat{\pi}$ (2.16) şi $\dfrac{\partial\hat{\pi}}{\partial t}$ (2.19) obţinem:

$$\frac{\partial^2\hat{\varphi}(x)}{\partial t^2}=\frac{\partial\hat{\pi}(x)}{\partial t}=\left(\hat{\nabla}^2-m^2\right)\hat{\varphi}(x)$$

adică, regăsim ecuaţia Klein-Gordon din mecanica cuantică:

$$\left(\frac{\partial^2}{\partial t^2}-\hat{\nabla}^2+m^2\right)\hat{\varphi}(x)=0$$

2.1.4 Operatorii de câmp scalar (Klein-Gordon) $\hat{\alpha}_k$ şi $\hat{\alpha}_k^\dagger$

(vezi 41-Cuantificare câmpuri clasice (19)-(23))

- φ ca soluţia generală a ec. Klein-Gordon, scrisă ca operator este:

$$\boxed{\hat{\varphi}(x)=\int d^3k\left(\hat{\alpha}_k e^{-i\,k\cdot x}+\hat{\alpha}_k^\dagger e^{i\,k\cdot x}\right) \quad \text{unde: } k\cdot x=\omega t-\vec{k}\cdot\vec{x}} \tag{2.20}$$

- Pentru a găsi operatorii $\hat{\alpha}_k$ şi $\hat{\alpha}_k^\dagger$, folosim câmpurile $\hat{\varphi}(x)$ şi $\hat{\dot{\varphi}}(x)$ la acelaşi timp $t_0=0$:

$$\begin{cases}\hat{\varphi}(\vec{x},0)=\int d^3k\left(\hat{\alpha}_k\, e^{i\,\vec{k}\cdot\vec{x}}+\hat{\alpha}_k^\dagger\, e^{-i\,\vec{k}\cdot\vec{x}}\right)\\ \hat{\dot{\varphi}}(\vec{x},0)=\int d^3k(-i\,\omega_k)\left(\hat{\alpha}_k\, e^{i\,\vec{k}\cdot\vec{x}}-\hat{\alpha}_k^\dagger\, e^{-i\,\vec{k}\cdot\vec{x}}\right)\end{cases} \tag{2.21}$$

şi funcţia $\delta^{(3)}$ scrisă sub forma,

$$\int \frac{d^3\vec{x}}{(2\pi)^3}\, e^{-i\,\vec{k}\cdot\vec{x}}=\delta^{(3)}(\vec{k}) \tag{2.22}$$

- Separăm operatorii din (2.21) cu $\delta^3(\vec{k})$ (2.22), prin integrare tip $\int d^3k\, f(\vec{k})\,\delta^3(\vec{k})=f(0)$, deci înmulţim (2.21) cu (2.22)

$$\begin{cases}\int \frac{d^3\vec{x}}{(2\pi)^3}\hat{\varphi}(\vec{x},0)\, e^{-i\,\vec{k}\cdot\vec{x}}=\hat{\alpha}_k+\hat{\alpha}^\dagger_{-k}\\ \int \frac{d^3\vec{x}}{(2\pi)^3}\hat{\dot{\varphi}}(\vec{x},0)\, e^{-i\,\vec{k}\cdot\vec{x}}=(-i\,\omega_k)\left(\hat{\alpha}_k-\hat{\alpha}^\dagger_{-k}\right)\end{cases} \tag{2.23}$$

- de unde

$$\begin{cases}\hat{\alpha}_k=\frac{1}{2}\int \frac{d^3\vec{x}}{(2\pi)^3}\left(\hat{\varphi}(\vec{x},0)+\frac{i}{\omega_k}\hat{\dot{\varphi}}(\vec{x},0)\right)e^{-i\,\vec{k}\cdot\vec{x}}\\ \hat{\alpha}_k^\dagger=\frac{1}{2}\int \frac{d^3\vec{x}}{(2\pi)^3}\left(\hat{\varphi}(\vec{x},0)-\frac{i}{\omega_k}\hat{\dot{\varphi}}(\vec{x},0)\right)e^{i\,\vec{k}\cdot\vec{x}}\end{cases} \tag{2.24}$$

-

2.2 Soluţii de câmp cuantic scalar (Klein-Gordon) $\hat{\varphi}$ şi $\hat{\pi}$

(vezi 41-Cuantificare câmpuri clasice (24)-(26))

- Cu $\hat{\alpha}_k$ şi $\hat{\alpha}_{k'}^\dagger$ (2.24), folosind $\hat{\pi}=\hat{\dot{\varphi}}$ (2.16) şi rel. de comutare (2.14) la acelaşi t (câmpurile evaluate la poziţii $\vec{x}$ şi $\vec{x}'$ diferite, dar la acelaşi timp $t=t'=0$) putem calcula $\left[\hat{\alpha}_k,\hat{\alpha}_{k'}^\dagger\right]$:

$$\begin{aligned}\left[\hat{\alpha}_k,\hat{\alpha}_{k'}^\dagger\right]&=-\frac{1}{4}\int\frac{d^3\vec{x}\,d^3x'}{(2\pi)^6}\left\{\frac{i}{\omega_{k'}}\left[\hat{\varphi}(\vec{x},0),\hat{\dot{\varphi}}(\vec{x'},0)\right]+\frac{i}{\omega_k}\left[\hat{\varphi}(\vec{x'},0),\hat{\dot{\varphi}}(\vec{x},0)\right]\right\}e^{-i\,\vec{k}\cdot\vec{x}+i\,\vec{k}'\cdot\vec{x'}}\\ &=-\frac{1}{4}\int\frac{d^3\vec{x}\,d^3x'}{(2\pi)^6}\left\{\frac{i}{\omega_{k'}}\left[i\,\delta^{(3)}(\vec{x}-\vec{x'})\right]+\frac{i}{\omega_k}\left[i\,\delta^{(3)}(\vec{x}-\vec{x'})\right]\right\}e^{-i\,\vec{k}\cdot\vec{x}+i\,\vec{k}'\cdot\vec{x'}}\\ &=\frac{1}{4}\int\frac{d^3\vec{x}}{(2\pi)^6}\left\{\frac{1}{\omega_{k'}}+\frac{1}{\omega_k}\right\}e^{-i\,(\vec{k}-\vec{k}')\cdot\vec{x}}=\frac{1}{(2\pi)^3\,2\omega_k}\delta^{(3)}\left(\vec{k}-\vec{k}'\right)\end{aligned}$$

- Definim operatorii de creare $\boxed{\hat{a}_k^\dagger=\hat{\alpha}_k^\dagger\sqrt{(2\pi)^3\,2\omega_k}}$ şi anihilare $\boxed{\hat{a}_k=\hat{\alpha}_k\sqrt{(2\pi)^3\,2\omega_k}}$ atunci relaţiile de comutare între operatorii de creare şi anihilare sunt:

$$\left[\hat{a}_k,\hat{a}_{k'}^\dagger\right]=\delta^{(3)}\left(\vec{k}-\vec{k}'\right) \tag{2.25}$$

- Am regăsit prin (2.25) relaţiile de comutare $\left[\hat{a},\hat{a}^\dagger\right]=1$ (2.4) pt. operatorii de oscilator armonic cuantic, de data asta pentru operatorii de câmp:

- Inlocuind în (2.20) $\hat{\alpha}_k=\frac{\hat{a}_k}{\sqrt{(2\pi)^3\,2\omega_k}}$ şi $\hat{\alpha}_k^\dagger=\frac{\hat{a}_k^\dagger}{\sqrt{(2\pi)^3\,2\omega_k}}$, operatorii $\hat{\varphi}(x)$ şi $\hat{\pi}(x)$ devin

$$\hat{\varphi}(x)=\int\frac{d^3k}{\sqrt{(2\pi)^3\,2\omega_k}}\left(\hat{a}_k\, e^{-i\,k\cdot x}+\hat{a}_k^\dagger\, e^{i\,k\cdot x}\right)\quad \text{unde}\quad k\cdot x=\omega t-\vec{k}\cdot\vec{x} \tag{2.26}$$

$$\hat{\pi}(x)=\hat{\dot{\varphi}}(x)=\int\frac{d^3k}{\sqrt{(2\pi)^3\,2\omega_k}}(-i\omega_k)\left(\hat{a}_k\, e^{-i\,k\cdot x}-\hat{a}_k^\dagger\, e^{i\,k\cdot x}\right) \tag{2.27}$$

-

2.2.1 Soluţii discrete şi soluţii continui de câmp cuantic scalar

- Reluăm Lagrangian-ul de câmp liber relativist (Klein-Gordon) scris acum pentru operatori de câmp cuantic:

$$\mathcal{L}(\hat\varphi,\dot{\hat\varphi})=\frac{1}{2}\Big(\dot{\hat\varphi}^\dagger\dot{\hat\varphi}-\nabla\hat\varphi^\dagger\cdot\nabla\hat\varphi-m^2\hat\varphi^\dagger\hat\varphi\Big)=\frac{1}{2}\Big(\partial_\mu\hat\varphi^\dagger\partial^\mu\hat\varphi-m^2\hat\varphi^\dagger\hat\varphi\Big)$$

- Ecuaţia de mişcare Euler-Lagrange (*ecuaţia de câmp*)

$$\frac{\partial\mathcal{L}}{\partial\hat\varphi}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\hat\varphi}}-\nabla\frac{\partial\mathcal{L}}{\partial\left(\nabla\hat\varphi\right)}=0$$

- Conduce la ecuaţia de câmp K-G real $\hat\varphi$ $\boxed{\left(\partial_t^2-\nabla^2+m^2\right)\hat\varphi=0}$ sau $\boxed{\left(\partial_\mu\partial^\mu+m^2\right)\hat\varphi=0}$ (2.28)
- Similar pt. ecuaţia de câmp K-G complex $\hat\varphi^\dagger$ $\boxed{\left(\partial_t^2-\nabla^2+m^2\right)\hat\varphi^\dagger=0}$ sau $\boxed{\left(\partial_\mu\partial^\mu+m^2\right)\hat\varphi^\dagger=0}$ (2.29)
- soluţii discrete de unde plane ortonormate (cu constrângeri de volum) — soluţii continui de unde plane ortonormate (fără constrângeri de volum)

$$\begin{cases}\hat\varphi(x)=\sum\limits_{\vec k}\dfrac{1}{\sqrt{2V\omega_{\vec k}}}\left(\hat a(\vec k)\,e^{-ik\cdot x}+\hat b^\dagger(\vec k)\,e^{ik\cdot x}\right)\\[2mm] \hat\varphi^\dagger(x)=\sum\limits_{\vec k}\dfrac{1}{\sqrt{2V\omega_{\vec k}}}\left(\hat b(\vec k)\,e^{-ik\cdot x}+\hat a^\dagger(\vec k)\,e^{ik\cdot x}\right)\end{cases}\qquad \begin{cases}\hat\varphi(x)=\displaystyle\int\frac{d^3\vec k}{\sqrt{(2\pi)^3 2\omega_{\vec k}}}\left(\hat a(\vec k)\,e^{-ik\cdot x}+\hat b^\dagger(\vec k)e^{ik\cdot x}\right)\\[2mm] \hat\varphi^\dagger(x)=\displaystyle\int\frac{d^3\vec k}{\sqrt{(2\pi)^3 2\omega_{\vec k}}}\left(\hat b(\vec k)\,e^{-ik\cdot x}+\hat a^\dagger(\vec k)e^{ik\cdot x}\right)\end{cases}$$

- Comutatorii coeficienţilor de stări discrete sunt (2.4), iar cei de stări continui sunt (2.25):

$$\Big[\hat a(\vec k),\hat a^\dagger(\vec k')\Big]=\Big[\hat b(\vec k),\hat b^\dagger(\vec k')\Big]=\delta_{\vec k\vec k'}\quad(2.4)\qquad \Big[\hat a(\vec k),\hat a^\dagger(\vec k')\Big]=\Big[\hat b(\vec k),\hat b^\dagger(\vec k')\Big]=\delta(\vec k-\vec k')\quad(2.25)$$

-

2.3 Hamiltonian de câmp cuantic scalar - soluţii discrete

- Acum avem două componente de câmp $\hat\varphi$ şi $\hat\varphi^\dagger$, şi două câmpuri conjugate $\hat\pi$ şi $\hat\pi^\dagger$.

Hamiltonian-ul de câmp scalar relativist (Klein-Gordon) scris acum prin operatorii de câmp cuantic: Cu Lagrangianul $\mathcal{L}=\partial_\mu\hat\varphi^\dagger\partial^\mu\hat\varphi-m^2\hat\varphi^\dagger\hat\varphi$ avem

$$\mathcal{H}(\hat\varphi,\hat\pi)=\hat\pi\dot{\hat\varphi}+\hat\pi^\dagger\dot{\hat\varphi}^\dagger-\mathcal{L}=\Big(\dot{\hat\varphi}\dot{\hat\varphi}^\dagger+\nabla\hat\varphi^\dagger\cdot\nabla\hat\varphi+m^2\hat\varphi^\dagger\hat\varphi\Big)$$

- Să evaluăm Hamiltonian-ul total, exprimat prin <u>componentele discrete</u> $\hat\varphi$ şi $\hat\varphi^\dagger$:

$$H=\int\mathcal{H}d^3x=\int\Big(\dot{\hat\varphi}\dot{\hat\varphi}^\dagger+\nabla\hat\varphi^\dagger\cdot\nabla\hat\varphi+m^2\hat\varphi^\dagger\hat\varphi\Big)d^3x=$$

$$=\int\Bigg(\sum_{\vec k}\frac{\partial}{\partial t}\overbrace{\frac{1}{\sqrt{2V\omega_{\vec k}}}\Big[\hat a(\vec k)e^{-ikx}+\hat b^\dagger(\vec k)e^{ikx}\Big]}^{\hat\varphi}\Bigg)\Bigg(\sum_{\vec k'}\frac{\partial}{\partial t}\overbrace{\frac{1}{\sqrt{2V\omega_{\vec k'}}}\Big[\hat b(\vec k')e^{-ik'x}+\hat a^\dagger(\vec k')e^{ik'x}\Big]}^{\hat\varphi^\dagger}\Bigg)d^3\vec x$$

$$+\int\Big(-\partial_i\hat\varphi^\dagger\partial^i\hat\varphi+m^2\hat\varphi^\dagger\hat\varphi\Big)\,d^3\vec x\quad(2.30)$$

- Linia din mijloc, adică termenul $\int\dot{\hat\varphi}\dot{\hat\varphi}^\dagger d^3x$, după efectuarea derivatei temporale, devine:

$$\int\Bigg(\sum_{\vec k}\frac{i\omega_{\vec k}}{\sqrt{2V\omega_{\vec k}}}\Big[-\hat a(\vec k)e^{-ikx}+\hat b^\dagger(\vec k)e^{ikx}\Big]\Bigg)\Bigg(\sum_{\vec k'}\frac{i\omega_{\vec k'}}{\sqrt{2V\omega_{\vec k'}}}\Big[-\hat b(\vec k')e^{-ik'x}+\hat a^\dagger(\vec k')e^{ik'x}\Big]\Bigg)d^3\vec x$$

sau
$$\sum_{\vec k}\sum_{\vec k'}\left(\frac{-\sqrt{\omega_{\vec k}}\sqrt{\omega_{\vec k'}}}{2V}\int\begin{bmatrix}\hat a(\vec k)\,\hat b(\vec k')e^{-ik\cdot x}e^{-ik'\cdot x}-\hat a(\vec k)\,\hat a^\dagger(\vec k')e^{-ik\cdot x}e^{ik'\cdot x}\\-\hat b^\dagger(\vec k)\,\hat b(\vec k')e^{ik\cdot x}e^{-ik'\cdot x}+\hat b^\dagger(\vec k)\,\hat a^\dagger(\vec k')e^{ik\cdot x}e^{ik'\cdot x}\end{bmatrix}d^3x\right)$$

-

2.3.1 Hamiltonian de câmp cuantic scalar - soluţii discrete (cont.)

- Iniţial luăm primul termen din integrala (2.30), $\int \hat{\dot\varphi}\hat{\dot\varphi}^\dagger d^3x = \sum_{\vec k}\sum_{\vec k'}\left(\frac{\sqrt{\omega_{\vec k}}\sqrt{\omega_{\vec k'}}}{2V}\int\begin{bmatrix}-\hat a(\vec k)\,\hat b(\vec k')e^{-ik\cdot x}e^{-ik'\cdot x} + \hat a(\vec k)\,\hat a^\dagger(\vec k')e^{-ik\cdot x}e^{ik'\cdot x}\\ +\hat b^\dagger(\vec k)\,\hat b(\vec k')e^{ik\cdot x}e^{-ik'\cdot x} - \hat b^\dagger(\vec k)\,\hat a^\dagger(\vec k')e^{ik\cdot x}e^{ik'\cdot x}\end{bmatrix} d\right.$

- Sumele după $\vec k$ şi $\vec k'$ se iau de la $-\infty$ la $+\infty$ (pe cele 3 direcţii x, y şi z).
  Toţi termenii prin integrare (produse de funcţii proprii ortonormate) între limitele de graniţă, sunt nuli, cu excepţia celor cu $\vec k' = \vec k$ sau $\vec k' = -\vec k$. Explicităm $k\cdot x = \omega t - \vec k\cdot\vec x$,

$$\text{pt. } \vec k' = \vec k \quad \begin{bmatrix} -\hat a(\vec k)\,\hat b(\vec k)e^{-i(\omega_{\vec k}t-\vec k\cdot\vec x)}e^{-i(\omega_{\vec k}t-\vec k\cdot\vec x)} + \hat a(\vec k)\,\hat a^\dagger(\vec k)\,\underbrace{e^{-i(\omega_{\vec k}t-\vec k\cdot\vec x)}e^{i(\omega_{\vec k}t-\vec k\cdot\vec x)}}_{=1} \\ +\hat b^\dagger(\vec k)\,\hat b(\vec k)\,\overbrace{e^{i(\omega_{\vec k}t-\vec k\cdot\vec x)}\,e^{-i(\omega_{\vec k}t-\vec k\cdot\vec x)}}^{=1} - \hat b^\dagger(\vec k)\,\hat a^\dagger(\vec k)e^{i(\omega_{\vec k}t-\vec k\cdot\vec x)}e^{i(\omega_{\vec k}t-\vec k\cdot\vec x)} \end{bmatrix} +$$

$$\text{pt. } \vec k' = -\vec k \; + \begin{bmatrix} -\hat a(\vec k)\,\hat b(-\vec k)e^{-i(\omega_{\vec k}t-\vec k\cdot\cancel{\vec x})}e^{-i(\omega_{\vec k}t+\vec k\cdot\cancel{\vec x})} + \hat a(\vec k)\,\hat a^\dagger(-\vec k)e^{-i(\cancel{\omega_{\vec k}}t-\vec k\cdot\vec x)}e^{i(\cancel{\omega_{\vec k}}t+\vec k\cdot\vec x)} \\ +\hat b^\dagger(\vec k)\,\hat b(-\vec k)e^{i(\omega_{\vec k}\cancel{t}-\vec k\cdot\vec x)}e^{-i(\omega_{\vec k}\cancel{t}+\vec k\cdot\vec x)} - \hat b^\dagger(\vec k)\,\hat a^\dagger(-\vec k)e^{i(\omega_{\vec k}t-\vec k\cdot\cancel{\vec x})}e^{i(\omega_{\vec k}t+\vec k\cdot\cancel{\vec x})} \end{bmatrix} =$$

$$= \begin{bmatrix} -\hat a(\vec k)\,\hat b(-\vec k)\,e^{-2i\omega_{\vec k}t} + \hat a(\vec k)\,\hat a^\dagger(\vec k) + \hat b^\dagger(\vec k)\,\hat b(\vec k) - \hat b^\dagger(\vec k)\,\hat a^\dagger(-\vec k)\,e^{2i\omega_{\vec k}t} \\ -\hat a(\vec k)\hat b(\vec k)e^{-2i\omega_{\vec k}t}e^{2i\vec k\cdot\vec x} - \hat b^\dagger(\vec k)\hat a^\dagger(\vec k)e^{2i\omega_{\vec k}t}e^{-2i\vec k\cdot\vec x} + \hat a(\vec k)\hat a^\dagger(-\vec k)e^{2i\vec k\cdot\vec x} + \hat b^\dagger(\vec k)\hat b(-\vec k)e^{-2i\vec k\cdot\vec x} \end{bmatrix}$$

- Termenii marcaţi reprezintă produse scalare de *funcţii proprii ortogonale* (staţionare), adică integrarea lor pe sp. de coordonate între limitele de graniţă dau zero, sunt de tip $\int \sin\alpha x\, \sin\beta x\, dx$ (vezi stări cuantice pe cutie). Rămân doar cei nemarcaţi.

-

2.3.2 Hamiltonian de câmp cuantic scalar - soluţii discrete (cont.)

- Deci în integrala după primul termen (2.30) rămâne,

$$\begin{aligned}\int \hat{\dot\varphi}\hat{\dot\varphi}^\dagger d^3x &= \sum_{\vec k}\frac{\omega_{\vec k}}{2}\left(-\hat a(\vec k)\hat b(-\vec k)e^{-2i\omega_k t} + \hat a(\vec k)\hat a^\dagger(\vec k) + \hat b^\dagger(\vec k)\hat b(\vec k) - \hat b^\dagger(\vec k)\hat a^\dagger(-\vec k)e^{2i\omega_k t}\right)\\ &= \sum_{\vec k}\frac{(\omega_{\vec k})^2}{2\omega_{\vec k}}\left(-\hat a(-\vec k)\hat b(\vec k)e^{-2i\omega_k t} + \hat a(\vec k)\hat a^\dagger(\vec k) + \hat b^\dagger(\vec k)\hat b(\vec k) - \hat b^\dagger(-\vec k)\hat a^\dagger(\vec k)e^{2i\omega_k t}\right)\end{aligned} \tag{2.31}$$

  Schimbarea de semn pt. $\vec k$ în I-ul şi al IV-lea termen s-a făcut pt. uz ulterior, deoarece sumarea se face după toate val. $\vec k$ (pozitive şi negative), iar pt. orice $\vec k$ avem alta cu $-\vec k$.

- Un calcul similar pentru termenul următor (cu derivatele spaţiale) din (2.30) ne dă:

$$-\int \partial_i\hat\varphi^\dagger\partial^i\hat\varphi d^3x = \int \partial_i\hat\varphi^\dagger\partial_i\hat\varphi d^3x =$$

$$= \int\left(\sum_{\vec k}\frac{ik_i}{\sqrt{2V\omega_k}}\left[\hat b(\vec k)e^{-ik\cdot x} - \hat a^\dagger(\vec k)e^{ik\cdot x}\right]\right)\left(\sum_{\vec k'}\frac{ik'_i}{\sqrt{2V\omega_{k'}}}\left[\hat a(\vec k')e^{-ik'\cdot x} - \hat b^\dagger(\vec k')e^{ik'\cdot x}\right]\right)d^3x$$

$$= \sum_{\vec k}\frac{\vec k^2}{2\omega_k}\left(\hat b(\vec k)\hat a(-\vec k)e^{-2i\omega_k t} + \hat a^\dagger(\vec k)\hat a(\vec k) + \hat b(\vec k)\hat b^\dagger(\vec k) + \hat a^\dagger(\vec k)\hat b^\dagger(-\vec k)e^{2i\omega_k t}\right) \tag{2.32}$$

  Notă: atât termenii funcţie de $\vec k$ cât şi de $-\vec k$ au semnul schimbat, deoarece $k_i = -k'_i$.

- Similar, pentru termenul de masă din (2.30) obţinem: $\int m^2\hat\varphi^\dagger\hat\varphi d^3x =$

$$= \int m^2\left(\sum_{\vec k}\frac{1}{\sqrt{2V\omega_k}}\left[\hat b(\vec k)e^{-ik\cdot x} + \hat a^\dagger(\vec k)e^{ik\cdot x}\right]\right)\left(\sum_{\vec k'}\frac{1}{\sqrt{2V\omega_{k'}}}\left[\hat a(\vec k')e^{-ik'\cdot x} + \hat b^\dagger(\vec k')e^{ik'\cdot x}\right]\right)d^3x$$

$$= \sum_{\vec k}\frac{m^2}{2\omega_k}\left(\hat b(\vec k)\hat a(-\vec k)e^{-2i\omega_k t} + \hat b(\vec k)\hat b^\dagger(\vec k) + \hat a^\dagger(\vec k)\hat a(\vec k) + \hat a^\dagger(\vec k)\hat b^\dagger(-\vec k)e^{2i\omega_k t}\right) \tag{2.33}$$

2.3.3 Hamiltonian prin operator număr de particule - sol. discrete

- Adunând rezultatele (2.31), (2.32) şi (2.33), şi folosind relaţia $\vec{k}^2+m^2=(\omega_k)^2$ precum şi relaţiile de comutare a coeficienţilor:

$$\left[\hat{a}(\vec{k}),\hat{a}^\dagger(\vec{k}')\right]=\left[\hat{b}(\vec{k}),\hat{b}^\dagger(\vec{k}')\right]=\delta_{\vec{k}\vec{k}'}(\text{sol. discrete});\quad =\delta(\vec{k}-\vec{k}')(\text{sol. continui}) \tag{2.34}$$

- Hamiltonianul total de câmp cuantic scalar pentru stările de soluţii discrete, este

$$\begin{aligned}\hat{H}&=\sum_{\vec{k}}\frac{\omega_k}{2}\Big(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k})+\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})+\hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})+\hat{b}(\vec{k})\hat{b}^\dagger(\vec{k})\Big)\\ &=\sum_{\vec{k}}\omega_k\left(\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})+\frac{1}{2}+\hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})+\frac{1}{2}\right)=\boxed{\sum_{\vec{k}}\omega_k\left(\hat{N}_a(\vec{k})+\frac{1}{2}+\hat{N}_b(\vec{k})+\frac{1}{2}\right)}\end{aligned} \tag{2.35}$$

- Valoarea proprie a Hamiltonian-ului (2.35), pentru o stare dată, este energia totală a stării respective. De exemplu, pentru o stare multiparticulă: 1 particulă cu energia ω_p, 2 particule cu energia ω_q şi 1 particulă cu energia ω_r, avem

$$E_{tot}\,|\varphi_p 2\varphi_q\varphi_r\rangle=(\omega_p+2\omega_q+\omega_r)\,|\varphi_p 2\varphi_q\varphi_r\rangle \qquad \text{adică, ecuaţia cu val. proprii}$$

$$\begin{aligned}\hat{H}\,|\varphi_p 2\varphi_q\varphi_r\rangle&=\sum_{\vec{k}}\omega_k\left(\hat{N}_a(\vec{k})+\frac{1}{2}+\hat{N}_b(\vec{k})+\frac{1}{2}\right)|\varphi_p 2\varphi_q\varphi_r\rangle\\ &=\Big(n_p\omega_p+n_q\omega_q+n_r\omega_r+(\text{1/2 energia stării de vacuum})\Big)|\varphi_p 2\varphi_q\varphi_r\rangle\end{aligned} \tag{2.36}$$

- In concluzie, operatorii $\hat{N}_a$ şi $\hat{N}_b$ sunt operatorii număr de particule.
De notat că dacă am fi folosit în (2.35) "ordonarea normală" (care presupune că operatorii comută, chiar dacă sunt ne-comutatori), nu am fi obţinut factorii $1/2$ de mai sus.

2.4 Hamiltonian de câmp cuantic scalar - soluţii continui

- Folosind soluţiile continui $\hat{\varphi}$ şi $\hat{\varphi}^\dagger$ (reamintim: $k\cdot x=\omega_{\vec{k}}\,t-\vec{k}\cdot\vec{x}$)

$$\begin{cases}\hat{\varphi}(x)=\displaystyle\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}\left(\hat{a}(\vec{k})\,e^{-ik\cdot x}+\hat{b}^\dagger(\vec{k})e^{ik\cdot x}\right)\\ \hat{\varphi}^\dagger(x)=\displaystyle\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}\left(\hat{b}(\vec{k})\,e^{-ik\cdot x}+\hat{a}^\dagger(\vec{k})e^{ik\cdot x}\right)\end{cases}$$

$$H=\int\mathcal{H}\,d^3x=\int\left(\dot{\hat{\varphi}}\dot{\hat{\varphi}}^\dagger+\nabla\hat{\varphi}^\dagger\cdot\nabla\hat{\varphi}+m^2\hat{\varphi}^\dagger\hat{\varphi}\right)d^3x= \quad \text{după derivarea temporală}$$

Hamiltonian-ul total, este:

$$=\int\left(\int\frac{d^3\vec{k}\;\;i\omega_{\vec{k}}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}\Big[-\hat{a}(\vec{k})e^{-ik\cdot x}+\hat{b}^\dagger(\vec{k})e^{ik\cdot x}\Big]\right)\left(\int\frac{d^3\vec{k}'\;\;i\omega_{\vec{k}'}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}'}}}\Big[-\hat{b}(\vec{k}')e^{-ik'\cdot x}+\hat{a}^\dagger(\vec{k}')e^{ik'\cdot x}\Big]\right)d^3\vec{x}$$

$$+\int\left(-\partial_i\hat{\varphi}^\dagger\partial^i\hat{\varphi}+m^2\hat{\varphi}^\dagger\hat{\varphi}\right)d^3\vec{x} \tag{2.37}$$

- Linia din mijloc, adică termenul $\int\dot{\hat{\varphi}}\dot{\hat{\varphi}}^\dagger d^3x$, devine:

$$\int\left[\int\left(\frac{1}{2(2\pi)^3}\int\frac{-\omega_{\vec{k}}\omega_{\vec{k}'}}{\sqrt{\omega_{\vec{k}}}\sqrt{\omega_{\vec{k}'}}}\begin{bmatrix}\hat{a}(\vec{k})\,\hat{b}(\vec{k}')e^{-ik\cdot x}e^{-ik'\cdot x}-\hat{a}(\vec{k})\,\hat{a}^\dagger(\vec{k}')e^{-ik\cdot x}e^{ik'\cdot x}\\ -\hat{b}^\dagger(\vec{k})\,\hat{b}(\vec{k}')e^{ik\cdot x}e^{-ik'\cdot x}+\hat{b}^\dagger(\vec{k})\,\hat{a}^\dagger(\vec{k}')e^{ik\cdot x}e^{ik'\cdot x}\end{bmatrix}d^3\vec{x}\right)d^3\vec{k}'\right]d^3\vec{k}$$

- Folosind: $\delta^3(\vec{k}-\vec{k}')=\frac{1}{(2\pi)^3}\int e^{(\vec{k}-\vec{k}')\cdot\vec{x}}d^3\vec{x}$ în integrarea după $d^3\vec{x}$, avem similar (2.31)

$$\int\frac{\omega_{\vec{k}}}{2}\left[-\hat{a}(\vec{k})\hat{b}(-\vec{k})e^{-2i\omega_k t}+\underbrace{\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k})}_{\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})+\delta(\vec{k}-\vec{k})}+\hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})-\hat{b}^\dagger(\vec{k})\hat{a}^\dagger(-\vec{k})e^{2i\omega_k t}\right]d^3\vec{k}$$

- Folosind relaţiile de comutare (2.34) pentru soluţii continui şi evaluând ultima linie din (2.37) similar cu cazul discret (2.32-2.33) obţinem o expresie similară (2.35)

$$\boxed{H=\int\omega_k\left(N_a(\vec{k})+\frac{1}{2}\delta(0)+N_b(\vec{k})+\frac{1}{2}\delta(0)\right)d^3\vec{k}} \tag{2.38}$$

2.4.1 Hamiltonian de câmp cuantic scalar - soluţii continui

- Observaţie: $\delta(0)$ reprezintă contribuţia energie de vacuum. Este infinită şi are unităţi de $1/k^3$ (inversul impulsului la puterea a 3-a). Deoarece $k = 2\pi/\lambda$, acesta are dimensiunea de volum (lungime la puterea a 3-a). Astfel, $\delta(0)$ reprezintă un volum infinit, întregul spaţiu. Atunci, densitatea de Hamiltonian, adică densitatea energiei de vacuum este

$$\mathcal{H}_{vac} = \frac{H_{vac}}{V} = \int \omega_{\vec{k}} \left(\frac{1}{2} + \frac{1}{2}\right) d^3\vec{k} \qquad \begin{pmatrix}\text{energia de vacuum din uni-}\\ \text{tatea de vol. din spaţiul fizic}\end{pmatrix} \tag{2.39}$$

Notă: Energia este $\hbar\omega$.
In (2.39) ca în toate relaţiile din QFT, folosim unităţi cu $\hbar = c = 1$.

2.5 Test

Slideuri suplimentare (de sarit)

De fapt, dacă interpretăm $\hat{a}_k$ şi $\hat{a}_k^\dagger$ ca operatori de creare şi anihilare, aceştia satisfac relaţia de comutare cu hamiltonian-ul

$$\left[\hat{H}, \hat{a}_k^\dagger\right] = \omega_k \hat{a}_k^\dagger \quad , \quad \left[\hat{H}, \hat{a}_k\right] = -\omega_k \hat{a}_k \tag{2.40}$$

astfel aceşti operatori conduc la crearea şi anihilarea de particule. In hamiltonian-ul (**??**), putem înlocui explicit câmpurile prin $\hat{a}_k^\dagger$ şi $\hat{a}_k$ şi folosind relaţia de comutare (2.25), obţinem expresia hamiltonian-ului în termeni de $\hat{a}_k^\dagger$ şi $\hat{a}_k$. După câteva operaţii algebrice, obţinem

$$\begin{aligned} H = \frac{1}{2}\int \frac{d^3k}{2\omega_k} & \left[\hat{a}_k\, \hat{a}_{-k} \exp(-2i\,\omega_k t) \left(-\omega_k^2 + \boldsymbol{k}^2 + m^2\right)\right. \\ & + \hat{a}_k^\dagger \hat{a}_k \left(\omega_k^2 + \boldsymbol{k}^2 + m^2\right) \\ & + \hat{a}_k \hat{a}_k^\dagger \left(\omega_k^2 + \boldsymbol{k}^2 + m^2\right) \\ & \left. + \hat{a}_k^\dagger \hat{a}_{-k}^\dagger \exp(2i\,\omega_k t) \left(-\omega_k^2 + \boldsymbol{k}^2 + m^2\right)\right] \end{aligned}$$

Deoarece $\omega_k^2 = \boldsymbol{k}^2 + m^2$, vom lăsa deoparte termenul dependent de timp şi obţinem

$$\hat{H} = \frac{1}{2}\int d^3k\, \omega_k \left[\hat{a}_k \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_k\right] \tag{2.41}$$

Acest rezultat este aproape identic cu cel obţinut mai demult,

$$\hat{H} = \frac{1}{2}\int d^3k\, \omega_k\, \hat{a}_k^\dagger \hat{a}_k \tag{2.42}$$

Efectuând comutarea între $\hat{a}_k$ şi $\hat{a}_k^\dagger$ din ecuaţia (2.41), obţinem

$$\hat{H} = \int d^3k\, \omega_k \left[\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\delta^{(3)}(0)\right] \tag{2.43}$$

Dar nu pare corect. Ce este acum $\delta^{(3)}(0)$? Să revenim la problema normării într-un spaţiu cub. Atunci

$$\hat{H} = \frac{1}{2}\sum_{\boldsymbol{k}} \omega_k \left[\hat{a}_k \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_k\right] = \sum_{\boldsymbol{k}} \left[\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right] \tag{2.44}$$

astfel $\delta^{(3)}(0)$ este o sumă infinită de energii de nivel zero. Energia fiecărei asemenea contribuţii este $\frac{1}{2}\omega_k$, şi nu este zero. Cum există o infinitate de asemenea contribuţii, obţinem o energie infinită în starea fundamentală.

Acest lucru însă nu crează probleme, deoarece nu atât valoarea absolută a energiei este importantă. Sens fizic au doar diferenţele de energie, care sunt finite. Deoarece întotdeauna se obţine o energie infinită, vom încerca să profităm de ocazie şi să înlăturăm această inadvertenţă pentru totdeauna. Pentru aceasta să amintim că energia minimă în cazul *oscilatorului armonic* rezultă în urma *ambiguităţii de ordonare*. Cuantificarea *oscilatorului armonic* o puteam face pornind de la Hamiltonian-ul clasic (**??**)

$$H_{OA} = \frac{\omega}{2}(q - i\,p)(q + i\,p) \tag{2.45}$$

Dacă p şi q sunt numere reale, acesta este identic cu hamiltonian-ul obişnuit $\frac{\omega}{2}\left(p^2 + q^2\right)$. Dacă însă p şi q sunt operatori, acesta devine

$$\hat{H}_{OA} = \omega\,\hat{a}^\dagger\hat{a} \tag{2.46}$$

în locul celui obişnuit (vezi Anexa **??** (**??**)) $\omega(a^\dagger a + 1/2)$.

Astfel, printr-o ordonare potrivită putem elimina energia infinită (nefizică) din starea fundamentală.

Pentru un set de câmpuri libere $\phi_1(x_1), \phi_2(x_2), \ldots, \phi_n(x(n)$, se defineşte *produsul normal ordonat*

$$:\phi_1(x_1)\ldots\phi_n(x_n): \tag{2.47}$$

ca un produs obişnuit, cu diferenţa că toţi operatorii de creare se plasează la stânga, iar operatorii de anihilare se plasează la dreapta. Deoarece operatorii de creare comută între ei, la fel şi cei de anihilare, această alegere specifică în mod unic ordinea de acţiune. Astfel, în loc de $\hat{H}$, putem folosi $:\hat{H}:$ iar energia infinită a stării fundamentale se va scrie

$$:\hat{H} := \int d^3k\,\omega_k\hat{a}_k^\dagger\hat{a}_k \tag{2.48}$$