Procese QED în câmpuri laser intense
tema 14

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”Verbum sapienti sat est”

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Capitol 1

De la mecanica clasică la câmpuri clasice şi câmpuri cuantice

- Trecerea de la mecanica clasică la mecanica cuantică (cuantificarea I) se face prin înlocuirea *mărimi fizice → operatori*

  Coordonata şi impulsul devin operatori $\hat{x} = x \quad ; \quad \hat{p} = -i\hbar \frac{\partial}{\partial x}$

  Operatorii $\hat{x}$ şi $\hat{p}$ satisfac *relaţia de comutare* $[\hat{x}, \hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x} = i\,\hbar$ (1.1)

  Exemplu: Operatorul Hamilton pentru oscilatorul armonic este:
  $$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}k\,\hat{x}^2 \tag{1.2}$$

- In mecanica clasică, descriem *poziţia* prin coordonatele $\vec{x}$. Printr-o transformare de coordonate (ex. rotaţia unei bare) are loc transformarea reciprocă a coordonatelor ($x \leftrightarrow y$).

  La fel, câmpurile $\vec{E}$ şi $\vec{B}$ de exemplu, se pot transforma reciproc ($\vec{B} \leftrightarrow \vec{E}$).

  Ecuaţiile de mişcare din mecanică ne dădeau variaţia coordonatei x în funcţie de timp t.

  Aceasta arată că putem trata asemănător componentele câmpurilor cu cele de coordonate.

  In teoria (clasică a) câmpului vom descrie mărimea urmărită *câmpul* $\varphi(x,t)$ în mod similar în funcţie de variabila independentă, care de data asta este 4-coordonata x^μ. Atunci,

- Trecerea de la mecanica clasică la câmpuri clasice se face prin înlocuirea:

  Coordonate $x_i \to$ componente de câmp φ_i
  $$x_i \longrightarrow \varphi_i(\boldsymbol{x}, t) \tag{1.3}$$

  Viteze $\dot{x}_i \to$ derivate de câmp $\partial_\mu \varphi_i$
  $$\frac{dx_i}{dt} \longrightarrow \frac{\partial \varphi_i(\boldsymbol{x}, t)}{\partial x^\mu} \tag{1.4}$$

- Trecerea de la câmpuri clasice la câmpuri cuantice (cuantificarea a II-a) se face prin înlocuirea: *câmpuri clasice → operatori de câmp.*

  Câmpul φ devine operator: $\varphi(x,t) \to \hat{\varphi}(x,t)$

1.1 Ecuaţiile Hamilton de câmpuri clasice

- Hamiltonian (densitate): $\mathcal{H}(\psi,\pi)=\pi\dot{\psi}-\mathcal{L}$
- Variaţia Hamiltonian-ului $\delta\mathcal{H}=\cancel{\pi\delta\dot{\psi}}+\dot{\psi}\delta\pi-\frac{\delta\mathcal{L}}{\delta\psi}\delta\psi-\cancel{\frac{\delta\mathcal{L}}{\delta\dot{\psi}}\delta\dot{\psi}}=\dot{\psi}\delta\pi-\dot{\pi}\delta\psi$
- Ecuaţiile canonice Hamilton (exprimate cu derivata funcţională) $\boxed{\dot{\psi}=\frac{\delta\mathcal{H}}{\delta\pi}}$; $\boxed{\dot{\pi}=-\frac{\delta\mathcal{H}}{\delta\psi}}$
- Verificare Ecuaţia Hamilton pentru Câmpuri Clasice Schrödinger

  Cu $\mathcal{L}$ pentru câmp Schrödinger $\mathcal{L}=i\hbar\psi^*\dot{\psi}-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi-V\,\psi^*\psi$

$$\pi=\frac{\delta\mathcal{L}}{\delta\dot{\psi}}=i\hbar\psi^* \quad\Longrightarrow\quad \psi^*=\frac{\pi}{i\hbar}$$

  atunci

$$\mathcal{H}=\pi\dot{\psi}-\mathcal{L}=\cancel{i\hbar\psi^*\dot{\psi}}-\cancel{i\hbar\psi^*\dot{\psi}}+\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi+V\psi^*\psi=-\frac{i\hbar}{2m}\nabla\pi\cdot\nabla\psi+\frac{V}{i\hbar}\pi\psi$$

- Folosind derivata funcţională

$$\begin{cases}\dot{\psi}=\frac{\delta\mathcal{H}}{\delta\pi}=\frac{\partial\mathcal{H}}{\partial\pi}-\nabla\left(\frac{\partial\mathcal{H}}{\partial(\nabla\pi)}\right)=\frac{V}{i\hbar}\psi+\frac{i\hbar}{2m}\nabla^2\psi\\ \dot{\pi}=-\frac{\delta\mathcal{H}}{\delta\psi}=-\frac{\partial\mathcal{H}}{\partial\psi}+\nabla\left(\frac{\partial\mathcal{H}}{\partial(\nabla\psi)}\right)=-\frac{V}{i\hbar}\pi-\frac{i\hbar}{2m}\nabla^2\pi\end{cases}$$

- In a II-a ec. folosim: $\pi=i\hbar\psi^*$ şi $\dot{\pi}=i\hbar\,\partial\psi^*/\partial t$, atunci regăsim

  ecuaţia Schrödinger reală: $i\hbar\frac{\partial\psi}{\partial t}=V\psi-\frac{\hbar^2}{2m}\nabla^2\psi$

  ecuaţia Schrödinger complexă: $-i\hbar\frac{\partial\psi^*}{\partial t}=V\psi^*-\frac{\hbar^2}{2m}\nabla^2\psi^*$

- Verificare Ecuaţia Hamilton pentru Câmpuri Clasice Klein-Gordon

  Cu $\mathcal{L}$ pentru câmp Klein-Gordon $\mathcal{L}=\frac{1}{2}\left[\dot{\varphi}^2-(\nabla\varphi)^2-m^2\varphi^2\right]\equiv\frac{1}{2}\left(\partial_\mu\varphi\,\partial^\mu\varphi-m^2\varphi^2\right)$

$$\pi=\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}=\dot{\varphi}$$

  atunci

$$\mathcal{H}=\pi\dot{\varphi}-\mathcal{L}=\dot{\varphi}^2-\frac{1}{2}\left(\dot{\varphi}^2-(\nabla\varphi)^2-m^2\varphi^2\right)=\frac{1}{2}\left(\pi^2+(\nabla\varphi)^2+m^2\varphi^2\right) \tag{1.5}$$

- Folosind derivata funcţională

$$\begin{cases}\dot{\varphi}=\frac{\delta\mathcal{H}}{\delta\pi}=\frac{\partial\mathcal{H}}{\partial\pi}=\pi\\ \dot{\pi}=-\frac{\delta\mathcal{H}}{\delta\varphi}=-\frac{\partial\mathcal{H}}{\partial\varphi}+\nabla\left(\frac{\partial\mathcal{H}}{\partial(\nabla\varphi)}\right)=-m^2\varphi+\nabla^2\varphi\end{cases}$$

  Cu $\pi=\dot{\varphi}$ şi $\dot{\pi}=\ddot{\varphi}\Longrightarrow$ ec. Klein-Gordon: $\ddot{\varphi}-\nabla^2\varphi+m^2\varphi=0$

1.2 Descrierea Heisenberg

Transfer evoluţie: *stare Schrödinger* $\longrightarrow$ *operator Heisenberg*

- Ecuaţia Schrödinger exprimă evoluţia ψ_S în t: $i\hbar\frac{\partial\psi_S(x,t)}{\partial t} = \hat{H}\psi_S(x,t)$

- Soluţia ecuaţiei Schrödinger este: $\psi_S(x,t) = e^{-i\hat{H}(t-t_0)\hbar}\,\psi_S(x,t_0)$

- Stările Heisenberg (independente de timp) se definesc ca stările Schrödinger la $t = t_0$ $\quad \psi_H(x) = \psi_S(x,t_0)$

- Observabile $\hat{A} \longrightarrow$ valori medii $\longrightarrow$ variaţie în t: $\langle\hat{A}(t)\rangle = \int \psi_S^*(x,t)\,\hat{A}\,\psi_S(x,t)\,dx$

- Variaţia în timp a valorilor medii

$$\frac{d\langle\hat{A}\rangle}{dt} = \int\left(\frac{\partial\psi_S^*}{\partial t}\hat{A}\psi_S + \psi_S^*\frac{\partial\hat{A}}{\partial t}\psi_S + \psi_S^*\hat{A}\frac{\partial\psi_S}{\partial t}\right)dx =$$

$$= \langle\frac{\partial\hat{A}}{\partial t}\rangle + \frac{1}{i\hbar}\int\psi_S^*\left(\hat{A}\hat{H} - \hat{H}\hat{A}\right)\psi_S dx = \langle\frac{\partial\hat{A}}{\partial t}\rangle + \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle$$

- Definiţie observabile Heisenberg $\quad \hat{A}_H = \langle\hat{A}\rangle \quad$ şi $\quad \frac{\partial\hat{A}_H}{\partial t} = 0$

- Ecuaţia Heisenberg: (evoluţia $\hat{A}_H$ în t) $\quad \boxed{i\hbar\frac{d\hat{A}_H}{dt} = \left[\hat{A}_H,\hat{H}\right]}$

1.3 Câmpuri scalare cuantice

- Trecerea de la câmpuri clasice la câmpuri cuantice (cuantificarea a II-a) se face prin înlocuirea câmpurilor şi a derivatelor de câmp cu operatorii corespunzători.

$$\begin{array}{lll} \varphi \longrightarrow \hat{\varphi} & \pi \longrightarrow \hat{\pi} & \mathcal{H}(\varphi,\pi) \longrightarrow \hat{\mathcal{H}}(\hat{\varphi},\hat{\pi}) \\ \varphi^* \longrightarrow \hat{\varphi}^\dagger & \pi^* \longrightarrow \hat{\pi}^\dagger & \Big\{ \varphi,\pi \Big\} \longrightarrow \frac{1}{i\hbar}\left[\hat{\varphi},\hat{\pi} \right] \end{array} \tag{1.6}$$

1.3.1 Cuantificarea câmpurilor scalare

- Cuantificarea (câmpurilor) înseamnă trecerea la operatori (de câmp).

- Câmpul φ ca soluţia generală de undă plană (**??**), scrisă ca operator, este:

$$\hat{\varphi}(x) = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3\, 2\omega_k}} \left(\hat{\alpha}_k e^{-i\,k\cdot x} + \hat{\alpha}_k^\dagger e^{i\,k\cdot x} \right) \qquad \text{unde: } k\cdot x = \omega_k t - \vec{k}\cdot\vec{x} \tag{1.7}$$

- Pentru a găsi operatorii $\hat{\alpha}_k$ şi $\hat{\alpha}_k^\dagger$, folosim

- operatorii de câmp $\hat{\varphi}(x)$ şi $\dot{\hat{\varphi}}(x)$ la acelaşi timp $t_0 = 0$:

$$\begin{cases} \hat{\varphi}(\vec{x},0) = \displaystyle\int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3\, 2\omega_k}} \left(\hat{\alpha}_k\, e^{i\,\vec{k}\cdot\vec{x}} + \hat{\alpha}_k^\dagger\, e^{-i\,\vec{k}\cdot\vec{x}} \right) \\ \dot{\hat{\varphi}}(\vec{x},0) = \displaystyle\int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3\, 2\omega_k}} (-i\,\omega_k) \left(\hat{\alpha}_k\, e^{i\,\vec{k}\cdot\vec{x}} - \hat{\alpha}_k^\dagger\, e^{-i\,\vec{k}\cdot\vec{x}} \right) \end{cases} \tag{1.8}$$

- şi funcţia δ exprimată ca

$$\int \frac{d^3\vec{x}}{(2\pi)^3}\, e^{\mp i\,\vec{k}\cdot\vec{x}} = \delta^{(3)}(\vec{k}) \tag{1.9}$$

- Separăm operatorii din (1.8) cu $\delta^3(\vec{k})$ (1.9), prin integrarea $\displaystyle\int d^3\vec{k}\, f(\vec{k})\, \delta^3(\vec{k}) = f(0)$,

deci înmulţim (1.8) cu $\delta^{(3)}(\vec{k})$ (1.9) rămâne $f(\vec{k}=0)$.

$$\begin{cases} \displaystyle\int \frac{d^3\vec{x}}{(2\pi)^3} e^{-i\,\vec{k}\cdot\vec{x}}\, \hat{\varphi}(\vec{x},0) = \frac{\hat{\alpha}_k + \hat{\alpha}^\dagger_{-k}}{\sqrt{(2\pi)^3\, 2\omega_k}} \\ \displaystyle\int \frac{d^3\vec{x}}{(2\pi)^3} e^{-i\,\vec{k}\cdot\vec{x}}\, \dot{\hat{\varphi}}(\vec{x},0) = (-i\,\omega_k) \frac{\hat{\alpha}_k - \hat{\alpha}^\dagger_{-k}}{\sqrt{(2\pi)^3\, 2\omega_k}} \end{cases} \tag{1.10}$$

- de unde

$$\begin{cases} \hat{\alpha}_k = \dfrac{2\omega_k}{2} \displaystyle\int \frac{d^3\vec{x}}{\sqrt{(2\pi)^3}} e^{-i\,\vec{k}\cdot\vec{x}} \left(\hat{\varphi}(\vec{x},0) + \frac{i}{\omega_k} \dot{\hat{\varphi}}(\vec{x},0) \right) \\ \hat{\alpha}_k^\dagger = \dfrac{2\omega_k}{2} \displaystyle\int \frac{d^3\vec{x}}{\sqrt{(2\pi)^3}} e^{i\,\vec{k}\cdot\vec{x}} \left(\hat{\varphi}(\vec{x},0) - \frac{i}{\omega_k} \dot{\hat{\varphi}}(\vec{x},0) \right) \end{cases} \tag{1.11}$$

1.3.2 Operatori de creare şi anihilare cuante de câmp

- Cu $\hat{\alpha}_k$ şi $\hat{\alpha}^\dagger_{k'}$ (1.11), ţinând cont că pentru un câmp scalar liber, $\hat{\pi}=\dot{\hat{\varphi}}$, calculăm relaţiile de comutare la acelaşi timp: $\left[\hat{\alpha}_k, \hat{\alpha}^\dagger_{k'}\right]$

(câmpurile sunt evaluate la poziţii $\vec{x}$ şi $\vec{x'}$ diferite, dar la acelaşi $t=t'=0$),

$$\frac{\left[\hat{\alpha}_k, \hat{\alpha}^\dagger_{k'}\right]}{2\omega_k 2\omega_{k'}} = -\frac{1}{4}\int\frac{d^3\vec{x}\, d^3\vec{x}'}{(2\pi)^3}\left\{\frac{i}{\omega_{k'}}\left[\hat{\varphi}(\vec{x},0), \dot{\hat{\varphi}}(\vec{x'},0)\right] + \frac{i}{\omega_k}\left[\hat{\varphi}(\vec{x'},0), \dot{\hat{\varphi}}(\vec{x},0)\right]\right\}e^{-i\,\vec{k}\cdot\vec{x}+i\,\vec{k}'\cdot\vec{x'}}$$

$$= -\frac{1}{4}\int\frac{d^3\vec{x}\, d^3\vec{x}'}{(2\pi)^3}\left\{\frac{i}{\omega_{k'}}\left[i\,\delta^{(3)}(\vec{x}-\vec{x'})\right] + \frac{i}{\omega_k}\left[i\,\delta^{(3)}(\vec{x}-\vec{x'})\right]\right\}e^{-i\,\vec{k}\cdot\vec{x}+i\,\vec{k}'\cdot\vec{x'}}$$

$$= \frac{1}{4}\int\frac{d^3\vec{x}}{(2\pi)^3}\left\{\frac{1}{\omega_{k'}} + \frac{1}{\omega_k}\right\}e^{-i\,(\vec{k}-\vec{k}')\cdot\vec{x}} = \frac{1}{2\omega_k}\delta^{(3)}\left(\vec{k}-\vec{k}'\right) \text{ unde am folosit (1.9)}$$

- Definim operatorii de creare $\boxed{\hat{a}^\dagger_k = \hat{\alpha}^\dagger_k\sqrt{\frac{1}{2\omega_k}}}$ şi anihilare $\boxed{\hat{a}_k = \hat{\alpha}_k\sqrt{\frac{1}{2\omega_k}}}$

  atunci relaţiile de comutare între operatorii de creare şi anihilare sunt:
  $$\left[\hat{a}_k, \hat{a}^\dagger_{k'}\right] = \delta^{(3)}\left(\vec{k}-\vec{k}'\right) \tag{1.12}$$

- Cu (1.12) am regăsit relaţiile de comutare $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ pentru operatorii de oscilator armonic cuantic:

- Inlocuind în (1.8) $\hat{\alpha}_k = \sqrt{2\omega_k}\,\hat{a}_k$ şi $\hat{\alpha}^\dagger_k = \sqrt{2\omega_k}\,\hat{a}^\dagger_k$ operatorii $\hat{\varphi}(\vec{x})$ şi $\hat{\pi}(\vec{x})$ pentru $t=0$, devin

$$\hat{\varphi}(\vec{x}) = \int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_k}}\left(\hat{a}_k\,e^{i\,\vec{k}\cdot\vec{x}} + \hat{a}^\dagger_k\,e^{-i\,\vec{k}\cdot\vec{x}}\right) \tag{1.13}$$

$$\hat{\pi}(\vec{x}) = \dot{\hat{\varphi}}(\vec{x}) = \int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_k}}(-i\omega_k)\left(\hat{a}_k\,e^{i\,\vec{k}\cdot\vec{x}} - \hat{a}^\dagger_k\,e^{-i\,\vec{k}\cdot\vec{x}}\right) \tag{1.14}$$

1.3.3 Cuantificarea energiei câmpului scalar liber

- Având Hamiltonian-ul de câmp scalar (1.5)

$$\hat{\mathcal{H}}=\frac{1}{2}\int d^3\vec{x}\left[\hat{\pi}^2+(\nabla\hat{\varphi})^2+m^2\hat{\varphi}^2\right] \tag{1.15}$$

- Folosind expresiile pentru $\hat{\varphi}$ (1.13) şi $\hat{\pi}$ (1.14)

$$\begin{cases}\hat{\varphi}(\vec{x})=\displaystyle\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_k}}\left(\hat{a}_k\,e^{i\,\vec{k}\cdot\vec{x}}+\hat{a}_k^\dagger\,e^{-i\,\vec{k}\cdot\vec{x}}\right) & (1.13)\\ \hat{\pi}(\vec{x})=\displaystyle\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_k}}(-i\omega_k)\left(\hat{a}_k\,e^{i\,\vec{k}\cdot\vec{x}}-\hat{a}_k^\dagger\,e^{-i\,\vec{k}\cdot\vec{x}}\right) & (1.14)\end{cases}$$

- Atunci, Hamiltonian-ul va fi:

$$\begin{aligned}\hat{\mathcal{H}}=\frac{1}{2}\int d^3\vec{x}\,\frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^3}\Bigg[&-\frac{\sqrt{\omega_k\omega_{k'}}}{2}\left(\hat{a}_k e^{i\,\vec{k}\cdot\vec{x}}-\hat{a}_k^\dagger e^{-i\,\vec{k}\cdot\vec{x}}\right)\left(\hat{a}_{k'}e^{i\,\vec{k}'\cdot\vec{x}}-\hat{a}_{k'}^\dagger e^{-i\,\vec{k}'\cdot\vec{x}}\right)+\\ &+\frac{(i\vec{k})(i\vec{k}')}{2\sqrt{\omega_k\omega_{k'}}}\left(\hat{a}_k e^{i\,\vec{k}\cdot\vec{x}}-\hat{a}_k^\dagger e^{-i\,\vec{k}\cdot\vec{x}}\right)\left(\hat{a}_{k'}e^{i\,\vec{k}'\cdot\vec{x}}-\hat{a}_{k'}^\dagger e^{-i\,\vec{k}'\cdot\vec{x}}\right)+\\ &+\frac{m^2}{2\sqrt{\omega_k\omega_{k'}}}\left(\hat{a}_k e^{i\,\vec{k}\cdot\vec{x}}+\hat{a}_k^\dagger e^{-i\,\vec{k}\cdot\vec{x}}\right)\left(\hat{a}_{k'}e^{i\,\vec{k}'\cdot\vec{x}}+\hat{a}_{k'}^\dagger e^{-i\,\vec{k}'\cdot\vec{x}}\right)\Bigg]\end{aligned} \tag{1.16}$$

- Prin integrare după $d^3\vec{x}$ rezultă $\delta^{(3)}(\vec{k}\mp\vec{k}')$, apoi integrare după $d^3\vec{k}$ (vezi Anexa pag.10)

$$\hat{\mathcal{H}}=\frac{1}{4}\int d^3\vec{k}\,\frac{1}{\omega_k}\left[\left(-\omega_k^2+\vec{k}^2+m^2\right)\left(\hat{a}_k\hat{a}_{-\vec{k}}+\hat{a}_k^\dagger\hat{a}_{-k}^\dagger\right)+\left(\omega_k^2+\vec{k}^2+m^2\right)\left(\hat{a}_k\hat{a}_k^\dagger+\hat{a}_k^\dagger\hat{a}_k\right)\right]$$

- Operatorul Hamilton de câmp scalar este:

$$\hat{\mathcal{H}}=\frac{1}{2}\int d^3\vec{k}\;\omega_k\left(\hat{a}_k\hat{a}_k^\dagger+\hat{a}_k^\dagger\hat{a}_k\right)\overset{(1.12)}{=}\int d^3\vec{k}\;\omega_k\left(\hat{a}_k^\dagger\hat{a}_k+\frac{1}{2}\right) \tag{1.17}$$

  similar cu Hamiltonian-ul de oscilator armonic cuantic (**??**).

- Deci, operatorul densitate de energie pentru mode-ul k de oscilaţie a unui câmp scalar liber, este (introducem constanta $\hbar$):

$$\hat{E}_k=\hbar\omega_k\left(\hat{a}_k^\dagger\hat{a}_k+\frac{1}{2}\right) \tag{1.18}$$

  similar cu energia oscilatorului armonic cuantic (**??**)

ANEXA

- Efectuăm produsele din integrala de oscilaţie de mode k şi apoi integrăm după $d^3\vec{x}$:

$$I_k=\int d^3\vec{k}'\int\frac{d^3\vec{x}}{(2\pi)^3}\Big[-\frac{\sqrt{\omega_k\omega_{k'}}}{2}\Big(\hat{a}_k\hat{a}_{k'}e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}-\hat{a}_k\hat{a}^\dagger_{k'}e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}-\hat{a}^\dagger_k\hat{a}_{k'}e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}+\hat{a}^\dagger_k\hat{a}^\dagger_{k'}e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}\Big)+$$
$$+\frac{1}{2\sqrt{\omega_k\omega_{k'}}}\Big(-\vec{k}\vec{k}'\hat{a}_k\hat{a}_{k'}e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}+\vec{k}\vec{k}'\hat{a}_k\hat{a}^\dagger_{k'}e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}-\vec{k}\vec{k}'\hat{a}^\dagger_k\hat{a}_{k'}e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}-\vec{k}\vec{k}'\hat{a}^\dagger_k\hat{a}^\dagger_{k'}e^{-i(\vec{k}+\vec{k}'\cdot\vec{x})}\Big)+$$
$$+\frac{m^2}{2\sqrt{\omega_k\omega_{k'}}}\Big(\hat{a}_k\hat{a}_{k'}e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}+\hat{a}_k\hat{a}^\dagger_k e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}+\hat{a}^\dagger_k\hat{a}_{k'}e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}+\hat{a}^\dagger_k\hat{a}^\dagger_{k'}e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}\Big)\Big]$$

- Identificând funcţia $\delta(k\mp k')=\frac{1}{2\pi}\int e^{i(k\mp k')x}dx$ (scrisă 3-dim) şi integrând $d^3\vec{x}$ cu

$$f(\vec{k})=\int\delta^{(3)}(\vec{k}\mp\vec{k}')f(\vec{k}')\,d^3\vec{k}'=\int\Big(\frac{1}{(2\pi)^3}\int e^{i(\vec{k}\mp\vec{k}')\cdot\vec{x}}d^3\vec{x}\Big)f(\vec{k}')d^3\vec{k}' \quad \text{cu} \begin{cases}\vec{k}'=\vec{k}\\ \vec{k}'=-\vec{k}\end{cases}$$

- avem

$$\begin{aligned}I_k=\Big[&-\frac{\sqrt{\omega_k\omega_{-k}}}{2}\hat{a}_k\hat{a}_{-k}+\frac{k^2}{2\sqrt{\omega_k\omega_{-k}}}\hat{a}_k\hat{a}_{-k}+\frac{m^2}{2\sqrt{\omega_k\omega_{-k}}}\hat{a}_k\hat{a}_{-k}\\ &+\frac{\sqrt{\omega_k\omega_k}}{2}\hat{a}_k\hat{a}^\dagger_k+\frac{k^2}{2\sqrt{\omega_k\omega_k}}\hat{a}_k\hat{a}^\dagger_k+\frac{m^2}{2\sqrt{\omega_k\omega_k}}\hat{a}_k\hat{a}^\dagger_k\\ &+\frac{\sqrt{\omega_k\omega_k}}{2}\hat{a}^\dagger_k\hat{a}_k+\frac{k^2}{2\sqrt{\omega_k\omega_k}}\hat{a}^\dagger_k\hat{a}_k+\frac{m^2}{2\sqrt{\omega_k\omega_k}}\hat{a}^\dagger_k\hat{a}_k\\ &-\frac{\sqrt{\omega_k\omega_{-k}}}{2}\hat{a}^\dagger_k\hat{a}^\dagger_{-k}+\frac{k^2}{2\sqrt{\omega_k\omega_{-k}}}\hat{a}^\dagger_k\hat{a}^\dagger_{-k}+\frac{m^2}{2\sqrt{\omega_k\omega_{-k}}}\hat{a}^\dagger_k\hat{a}^\dagger_{-k}\Big]\end{aligned}$$

- Grupăm termenii de oscilaţie de mode k din parantezele drepte:

$$\Big[\Big(-\frac{\sqrt{\omega_k\omega_{-k}}}{2}+\frac{k^2}{2\sqrt{\omega_k\omega_{-k}}}+\frac{m^2}{2\sqrt{\omega_k\omega_{-k}}}\Big)\Big(\hat{a}_k\hat{a}_{-k}+\hat{a}^\dagger_k\hat{a}^\dagger_{-k}\Big)+$$
$$+\Big(\frac{\sqrt{\omega_k\omega_k}}{2}+\frac{k^2}{2\sqrt{\omega_k\omega_k}}+\frac{m^2}{2\sqrt{\omega_k\omega_k}}\Big)\Big(\hat{a}_k\hat{a}^\dagger_k+\hat{a}^\dagger_k\hat{a}_k\Big)\Big]=$$

Deoarece $\omega_k=\omega_{-k}$, aducem la numitor comun şi folosim relaţia $\omega_k^2=k^2+m^2$:

$$=\frac{1}{2\omega_k}\Big[\underbrace{\big(-\omega_k^2+k^2+m^2\big)}_{=0}\big(\hat{a}_k\hat{a}_{-k}+\hat{a}^\dagger_k\hat{a}^\dagger_{-k}\big)+\underbrace{\big(\omega_k^2+k^2+m^2\big)}_{=2\omega_k^2}\big(\hat{a}_k\hat{a}^\dagger_k+\hat{a}^\dagger_k\hat{a}_k\big)\Big]=$$
$$=\omega_k\Big(\hat{a}_k\hat{a}^\dagger_k+\hat{a}^\dagger_k\hat{a}_k\Big)=\omega_k\Big(2\hat{a}^\dagger_k\hat{a}_k+1\Big) \text{ am folosit relaţia de comutare } \hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=1$$

- Atunci, operatorul Hamilton de câmp scalar va fi integrala din modurile de oscilaţie $\vec{k}$:

$$\hat{\mathcal{H}}=\frac{1}{2}\int d^3k\,I_k=\frac{1}{2}\int d^3k\;\omega_k\Big(\hat{a}_k\hat{a}^\dagger_k+\hat{a}^\dagger_k\hat{a}_k\Big)=\int d^3k\;\omega_k\Big(\hat{a}^\dagger_k\hat{a}_k+\frac{1}{2}\Big) \tag{1.19}$$

1.3.4 Câmpul electromagnetic clasic

- Am văzut (# Undele-Electromagnetice), din ecuaţiile Maxwell fără surse, obţinem ecuaţiile de propagare a componentelor $\vec{E}$ şi $\vec{B}$, sub forma ecuaţiei undelor:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} = 0 \quad ; \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{B} = 0 \tag{1.20}$$

- $\vec{E}$ şi $\vec{B}$ pot fi exprimate prin ϕ şi $\vec{A}$ (vezi Ec.Maxwell pentru potenţiale)

$$\vec{E} = -\vec{\nabla}\phi_{ext} - \frac{\partial\vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla}\times\vec{A} \tag{1.21}$$

- In vid pentru $\phi_{ext} = const.$, $\vec{A}$ satisface aceeaşi ecuaţie de propagare a undelor:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = 0 \tag{1.22}$$

- Soluţia este unda plană:

$$\boxed{\vec{A}(\vec{r},t) = \vec{A}_0\cos(\vec{k}\cdot\vec{r} - \omega t)} \tag{1.23}$$

- In absenţa surselor (în vid) şi pentru $\phi_{ext} = const.$ relaţiile (1.21) ne dau:

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = -\underbrace{\omega\vec{A}_0}_{\vec{E}_0}\sin(\vec{k}\cdot\vec{r} - \omega t) \quad ; \quad \vec{B} = \vec{\nabla}\times\vec{A} = \underbrace{\vec{k}\times\vec{A}_0}_{\vec{B}_0}\sin(\vec{k}\cdot\vec{r} - \omega t) \tag{1.24}$$

- Soluţiile (1.23) se pot exprima prin exponenţiale (la fel ca soluţiile K-G cu $m = 0$),

cu $\vec{A}_0 = \vec{\epsilon}A_0$, $\vec{\epsilon}$ vector unitar polarizare a $\vec{E}$

$$\boxed{\vec{A}(\vec{r},t) = \vec{\epsilon}\,A_0\left(a\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} + a^* e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right)} \tag{1.25}$$

- iar componentele $\vec{E}$ şi $\vec{B}$ de câmp EM:

$$\begin{cases} \vec{E}(\vec{r},t) = -\dfrac{\partial\vec{A}}{\partial t} = i\omega\vec{\epsilon}A_0\left(a\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} - a^* e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right) \\ \vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A} = i(\vec{k}\times\vec{\epsilon})A_0\left(a\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} - a^* e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right) \end{cases} \tag{1.26}$$

1.3.5 Energia câmpului electromagnetic clasic

- Densitatea de energie a câmpului EM este: (vezi Undele-Electromagnetice):

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \frac{\epsilon_0}{2}\left(E^2 + \underbrace{c^2 B^2}_{E^2}\right) = \epsilon_0|\vec{E}|^2 = \epsilon_0\,\omega^2\vec{A}_0^2\sin^2(\vec{k}\cdot\vec{r} - \omega t) \tag{1.27}$$

unde am folosit (1.24): $\vec{E} = -\omega\vec{A}_0\sin(\vec{k}\cdot\vec{r} - \omega t)$

- Pentru calculul energiei, trebuie să ţinem cont şi de cele două componente (grade de libertate) de polarizare ϵ_p ale $\vec{E}$, ca două oscilaţii independente, în plan normal la direcţia de mişcare. Acestea dublează energia câmpului EM.

- Astfel, energia câmpului EM într-un volum finit V este:
$$W = 2\int_V u\, d^3\vec{r} = 2\epsilon_0 \int_V |\vec{E}|^2\, d^3\vec{r} \tag{1.28}$$

- Deoarece $|\vec{E}| = E_0 = \omega A_0$ avem:
$$W = 2\epsilon_0\, \omega^2 A_0^2 V \tag{1.29}$$

- Cuantificarea energiei implică: (energie foton Planck: $W = h\nu$)
$$W = \hbar\omega \tag{1.30}$$

- Egalând ultimele două relaţii, putem exprima:
$$A_0 = \sqrt{\frac{\hbar}{2\epsilon_0 \omega V}} \tag{1.31}$$

- Soluţia $\vec{A}(\vec{r}, t)$ (1.25) devine:
$$\vec{A}(\vec{r}, t) = \vec{\epsilon}\sqrt{\frac{\hbar}{2\epsilon_0 \omega V}} \left(a\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} + a^* e^{-i(\vec{k}\cdot\vec{r} - \omega t)}\right) \tag{1.32}$$

1.3.6 Cuantificarea câmpului electromagnetic

- Pentru cuantificarea oscilatorului armonic am trecut la operatori: $x \to \hat{x}$ şi $p \to \hat{p}$ iar în hamiltonian $(x, p) \to (\hat{x}, \hat{p}) \to (\hat{a}, \hat{a}^\dagger)$. La fel, pentru cuantificarea câmp EM trecem la operatori de câmp $\psi(x) \to \hat{\psi}(x)$, adică vom înlocui $\vec{A} \to \hat{\vec{A}}$. Pentru două grade de polarizare independente $\vec{\epsilon}_\lambda$ şi un set de frecvenţe ω_k proprii, folosind (1.32) pentru $\vec{A}$,

$$\hat{\vec{A}}(\vec{r}, t) = \sum_{k,\lambda} \vec{\epsilon}_\lambda \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \left(\hat{a}_{k\lambda} e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} + \hat{a}^\dagger_{k\lambda} e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)}\right) \tag{1.33}$$

în termenul doi, a^*- *complex conjugat*, prin trecerea la operator, devine $\hat{a}^\dagger$- *hermitic conjugat*

- Componentele cuantificate $\vec{E}$ şi $\vec{B}$ (1.26) ale câmpului EM, devin operatori:

$$\begin{cases} \hat{\vec{E}}(\vec{r}, t) = -\dfrac{\partial \hat{\vec{A}}}{\partial t} = i \displaystyle\sum_{k,\lambda} \omega_k \vec{\epsilon}_\lambda \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \left(\hat{a}_{k\lambda} e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} - \hat{a}^\dagger_{k\lambda} e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)}\right) \\ \hat{\vec{B}}(\vec{r}, t) = \hat{\vec{\nabla}} \times \hat{\vec{A}} = i \displaystyle\sum_{k,\lambda} (\vec{k} \times \vec{\epsilon}_\lambda) \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \left(\hat{a}_{k\lambda} e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} - \hat{a}^\dagger_{k\lambda} e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)}\right) \end{cases} \tag{1.34}$$

- Rescriem (1.34) cu $\vec{u}_\ell = \dfrac{\vec{\epsilon}_\lambda}{\sqrt{2\epsilon_0 V}} e^{i\,\vec{k}\cdot\vec{r}}$ şi indicele mod de oscilaţie $\ell = (k, \lambda)$, adică,

$$\begin{cases} \hat{E}(\vec{r}, t) = i \displaystyle\sum_\ell \sqrt{\hbar\omega_\ell} \left[\hat{a}_\ell e^{-i\omega_\ell t}\, \vec{u}_\ell(\vec{r}) - \hat{a}^\dagger_\ell e^{i\omega_\ell t}\, \vec{u}^*_\ell(\vec{r})\right] \\ \hat{B}(\vec{r}, t) = i \displaystyle\sum_\ell \sqrt{\frac{\hbar c^2}{\omega_\ell}} \left[\hat{a}_\ell e^{-i\omega_\ell t} \vec{k}_\ell \times \vec{u}_\ell(\vec{r}) - \hat{a}^\dagger_\ell e^{i\omega_\ell t} \vec{k}_\ell \times \vec{u}^*_\ell(\vec{r})\right] \end{cases} \tag{1.35}$$

1.3.7 Cuantificarea energiei câmpului electromagnetic

- Operatorul Hamilton ca energia câmpului EM (vezi # Undele EM) $\quad \hat{H}_{field}=\frac{1}{2}\int_V\left(\epsilon_0\hat{E}^2+\frac{\hat{B}^2}{\mu_0}\right)d^3r$

  unde

$$\int_V d^3r\left|\hat{E}\right|^2=i\sum_{\ell,m}\hbar\sqrt{\omega_\ell\omega_m}\Big[-\hat{a}_\ell\hat{a}_m e^{-i(\omega_\ell t+\omega_m t)}\int d^3r\,\vec{u}_\ell(\vec{r})\cdot\vec{u}_m(\vec{r})$$
$$+\hat{a}_\ell^\dagger\hat{a}_m e^{-i(\omega_\ell t-\omega_m t)}\int d^3r\,\vec{u}_\ell^*(\vec{r})\cdot\vec{u}_m(\vec{r})\Big]+\Big[h.c.\Big]$$

- Folosind relaţiile de ortonormare: $\quad \int d^3r\,\vec{u}_\ell^*(\vec{r})\,\vec{u}_m(\vec{r})=\delta_{\ell,m}\quad;\quad\int d^3r\,\vec{u}_\ell(\vec{r})\,\vec{u}_m(\vec{r})=\delta_{-\ell,m}$

  într-adevăr, cu $u_\ell\sim e^{i\,k_\ell x}$ avem $\int_{-\infty}^{\infty}e^{i\,k_\ell x}e^{-i\,k_m x}dx=\int_{-\infty}^{\infty}e^{i\,(k_\ell-k_m)x}dx=2\pi\delta_{\ell,m}$

- Obţinem: $\int_V d^3r\left|\hat{E}\right|^2=\sum_\ell\hbar\omega_\ell\left[\hat{a}_\ell^\dagger\hat{a}_\ell+\hat{a}_\ell\hat{a}_\ell^\dagger-\hat{a}_\ell\hat{a}_{-\ell}e^{-2i\omega_\ell t}-\hat{a}_\ell^\dagger\hat{a}_{-\ell}^\dagger e^{2i\omega_\ell t}\right]$

  la fel

$$\int_V d^3r\left|\hat{B}\right|^2=i\sum_{\ell,m}\frac{\hbar c^2}{\sqrt{\omega_\ell\omega_m}}\Big[-\hat{a}_\ell\hat{a}_m e^{-i(\omega_\ell t+\omega_m t)}\int d^3r\,(\vec{k}_\ell\times\vec{u}_\ell)\cdot(\vec{k}_m\times\vec{u}_m)$$
$$+\hat{a}_\ell^\dagger\hat{a}_m e^{-i(\omega_\ell t-\omega_m t)}\int d^3r\,(\vec{k}_\ell\times\vec{u}_\ell^*)\cdot(\vec{k}_m\times\vec{u}_m)\Big]+\Big[h.c.\Big]$$

Cu identităţile $(\vec{k}_\ell\times\vec{u}_\ell)\cdot(\vec{k}_m\times\vec{u}_m)=(\vec{k}_\ell\cdot\vec{k}_m)(\vec{u}_\ell\cdot\vec{u}_m)$, şi $\vec{k}_\ell\cdot\vec{u}_\ell=0$, avem,

$$\int d^3r(\vec{k}_\ell\times\vec{u}_\ell)\cdot(\vec{k}_m\times\vec{u}_m)=-k_\ell^2\delta_{-\ell,m}\quad;\quad\int d^3r(\vec{k}_\ell\times\vec{u}_\ell^*)\cdot(\vec{k}_m\times\vec{u}_m)=k_\ell^2\delta_{\ell,m}$$

- Obţinem: $\int_V d^3r\left|\hat{B}\right|^2=\sum_\ell\hbar\omega_\ell\left[\hat{a}_\ell^\dagger\hat{a}_\ell+\hat{a}_\ell\hat{a}_\ell^\dagger+\hat{a}_\ell\hat{a}_{-\ell}e^{-2i\omega_\ell t}+\hat{a}_\ell^\dagger\hat{a}_{-\ell}^\dagger e^{2i\omega_\ell t}\right]$

- In final, expresia Hamiltonian-ului în funcţie de operatorii de creare şi anihilare:

$$\hat{H}=\frac{1}{2}\sum_{k\lambda}\hbar\omega_k\left(\hat{a}_{k\lambda}^\dagger\hat{a}_{k\lambda}+\hat{a}_{k\lambda}\hat{a}_{k\lambda}^\dagger\right)=\sum_{k\lambda}\hbar\omega_k\left(\hat{a}_{k\lambda}^\dagger\hat{a}_{k\lambda}+\frac{1}{2}\right)\tag{1.36}$$

- Hamiltonian-ul se poate scrie ca cel de oscilator armonic $\quad \hat{H}_{field}=\hbar\omega_k\left(\hat{a}_{k\lambda}^\dagger\hat{a}_{k\lambda}+\frac{1}{2}\right)=\hbar\omega\left(\hat{N}_{k\lambda}+\frac{1}{2}\right)\quad(1.37)$

- $\hat{a}_{k\lambda}^\dagger\hat{a}_{k\lambda}=\hat{N}_{k\lambda}$ este operatorul *număr de fotoni* de frecvenţă ω_k şi polarizare ϵ_λ, similar cu operatorul *numar de particule* de la oscilator armonic.