Procese QED în câmpuri laser intense
tema 16

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”Verbum sapienti sat est”

Cuprins

Capitol 1

Câmpuri clasice - Rezumat

Câmpuri clasice	Lagrangian	Ecuaţiile de mişcare (Euler-Lagrange) Ecuaţiile de câmp $\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\right) = 0$
Oscilator armonic	$L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} \quad ; \quad p = \frac{\partial L}{\partial \dot{x}}$	$m\ddot{x} + kx = 0$
Coardă oscilantă	$\mathcal{L} = \overbrace{\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2}^{T} - \overbrace{\frac{\rho\, v^2}{2}\left(\frac{\partial y}{\partial x}\right)^2}^{V}$	$\frac{\partial^2 y}{\partial t^2} - v^2\frac{\partial^2 y}{\partial x^2} = 0$
Câmp scalar	$\mathcal{L} = \overbrace{\frac{1}{2}\frac{1}{c^2}\left(\partial_t\varphi\right)^2}^{T} - \overbrace{\frac{1}{2}\left(\partial_x\varphi\right)^2}^{V}$ $\equiv \frac{1}{2}\left(\partial_\mu\varphi\right)^2$	$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0$ $\partial_\mu\partial^\mu\varphi = 0$
Câmp scalar real masiv (Klein-Gordon) (cu $\hbar = c = 1$)	$\mathcal{L} = \frac{1}{2}\left[(\partial_t\varphi)^2 - (\nabla\varphi)^2 - m^2\varphi^2\right]$ $\equiv \frac{1}{2}\left[(\partial_\mu\varphi)^2 - m^2\varphi^2\right]$	$(\partial_t^2 - \nabla^2 + m^2)\,\varphi = 0$ $(\partial_\mu\partial^\mu + m^2)\,\varphi = 0$
Câmp scalar complex masiv (Klein-Gordon) (cu $\hbar = c = 1$)	$\mathcal{L} = \partial_t\varphi^*\partial_t\varphi - \nabla\varphi^*\cdot\nabla\varphi - m^2\varphi^*\varphi$ $\equiv \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi$	$(\partial_t^2 - \nabla^2 + m^2)\,\varphi^* \equiv (\partial_\mu\partial^\mu + m^2)\,\varphi^* = 0$ $(\partial_t^2 - \nabla^2 + m^2)\,\varphi \equiv (\partial_\mu\partial^\mu + m^2)\,\varphi = 0$
Câmp Schrödinger	$\mathcal{L} = i\hbar\varphi^*\dot{\varphi} - \frac{\hbar^2}{2m}\nabla\varphi^*\cdot\nabla\varphi - V\varphi^*\varphi$	$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\varphi + V\,\varphi$ $-i\hbar\frac{\partial\varphi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\varphi^* + V\varphi^*$
Câmp Dirac	$\mathcal{L} = i\,\hbar\,\bar{\varphi}\,\gamma^\mu\partial_\mu\varphi - mc\,\bar{\varphi}\,\varphi$ unde $\bar{\varphi} = \varphi^\dagger\gamma^0$	$i\,\hbar\,\gamma^\mu\partial_\mu\varphi - mc\,\varphi = 0$ $i\,\hbar\,\partial_\mu\bar{\varphi}\,\gamma^\mu + m\,c\,\bar{\varphi} = 0$
Câmp Maxwell	$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu_0\, j_\mu A^\mu$ unde $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$	$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$ $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$
Câmp Proca	$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu$ $= -\frac{1}{2}\left[(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right]$ $+ \frac{m^2}{2}A^2$	$\partial_\nu F^{\mu\nu} + m^2 A^\mu = 0$ $(\partial_\nu\partial^\nu)\,A^\mu - \partial^\mu\,(\partial_\nu A^\nu) + m^2 A^\mu = 0$

1.1 Ecuaţiile de câmp electromagnetic (Maxwell)

- Lagrangianul de câmp electromagnetic este:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu_0 j_\mu A^\mu} \tag{1.1}$$

- Tensorul de câmp electromagnetic este:

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\equiv\frac{\partial A_\nu}{\partial x^\mu}-\frac{\partial A_\mu}{\partial x^\nu} \quad ; \quad F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\equiv\frac{\partial A^\nu}{\partial x_\mu}-\frac{\partial A^\mu}{\partial x_\nu} \tag{1.2}$$

$$F_{\mu\nu}=\begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{pmatrix} \quad ; \quad F^{\mu\nu}=\begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix} \tag{1.3}$$

- Folosim tensorii metrici $g_{\mu\nu}$ de coborîre indici $g_{\nu\alpha}\partial_\mu A^\alpha = \partial_\mu A_\nu$ şi $g^{\mu\nu}$ de ridicare indici $g^{\mu\beta}\partial_\beta A^\nu = \partial^\mu A^\nu$. Astfel, la o transformare termenii temporali nu schimbă de semn $A_0 = A^0$, iar cei spaţiali schimbă de semn $A_i = -A^i$.
  Atunci, ridicarea sau coborîrea simultană a perechilor de indici spaţiali (i,j) nu schimbă de semn: $\partial_i A_j = \partial^i A^j$, iar ridicarea sau coborîrea simultană a indicilor time-space (0,j), (i,0) schimbă de semn: $\partial_0 A_j = -\partial^0 A^j$ şi $\partial_i A_0 = -\partial^i A^0$.
  De aceea, ţinând cont şi de antisimetria tensorului $F_{\mu\nu}=-F_{\nu\mu}$ sau $F^{\mu\nu}=-F^{\nu\mu}$ avem,

$$-F_{ij}=F_{ji}=F^{ji}=-F^{ij} \qquad F_{0i}=-F_{i0}=F^{i0}=-F^{0i} \qquad (i,j=1,2,3)$$

- I-ul termen al Lagrangian-ului (1.1) conţine $F_{\mu\nu}F^{\mu\nu}$

$$F_{\mu\nu}F^{\mu\nu}=-\overbrace{2\left(E_1^2+E_2^2+E_3^2\right)/c^2}^{F_{0i}F^{0i}+F_{i0}F^{i0}}+\overbrace{2\left(B_3^2+B_2^2+B_1^2\right)}^{F_{ij}F^{ij}+F_{ji}F^{ji}}=2\left(B^2-\frac{E^2}{c^2}\right)$$

  Explicit

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= (F_{12}F^{12}+F_{21}F^{21})+(F_{23}F^{23}+F_{32}F^{32})+(F_{31}F^{31}+F_{13}F^{13})+ \\ &\quad +(F_{01}F^{01}+F_{10}F^{10})+(F_{02}F^{02}+F_{20}F^{20})+(F_{03}F^{03}+F_{30}F^{30})= \\ &= (F_{12}F_{12}+F_{12}F_{12})+(F_{23}F_{23}+F_{23}F_{23})+(F_{31}F_{31}+F_{31}F_{31})- \\ &\quad -(F_{01}F_{01}+F_{01}F_{01})-(F_{02}F_{02}+F_{02}F_{02})-(F_{03}F_{03}+F_{03}F_{03})= \\ &= 2\Big(\partial_1 A_2-\partial_2 A_1\Big)^2+2\Big(\partial_2 A_3-\partial_3 A_2\Big)^2+2\Big(\partial_3 A_1-\partial_1 A_3\Big)^2- \\ &\quad -2\Big(\partial_0 A_1-\partial_1 A_0\Big)^2-2\Big(\partial_0 A_2-\partial_2 A_0\Big)^2-2\Big(\partial_0 A_3-\partial_3 A_0\Big)^2 \end{aligned}$$

- Vom deduce ecuaţiile de câmp (Maxwell) din ecuaţiile Euler-Lagrange:

$$\boxed{\frac{\partial \mathcal{L}}{\partial A_\nu}-\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)=0} \tag{1.4}$$

- Căutăm întâi derivatele Lagrangian-ului $\dfrac{\partial \mathcal{L}}{\partial A_\nu}$ în ecuaţia Euler-Lagrange (1.4).

In Lagrangian-ul (1.1) avem $-\mu_0 j_\mu A^\mu$ cu produsul scalar

$$j_\mu A^\mu = j_0 A^0 - j_i A^i = j_0 A_0 + j_i A_i,$$

iar derivatele $\dfrac{\partial \mathcal{L}}{\partial A_\nu}$ sunt: $\dfrac{\partial \mathcal{L}}{\partial A_0} = -\mu_0 j_0 = -\mu_0 j^0$; $\dfrac{\partial \mathcal{L}}{\partial A_i} = -\mu_0 j_i = \mu_0 j^i$

Atunci I-ul termen al ecuaţiei Euler-Lagrange (1.4) $\Longrightarrow$ $$\frac{\partial \mathcal{L}}{\partial A_\nu} = -\mu_0 j^\nu \tag{1.5}$$

- Căutăm acum şi derivatele Lagrangian-ului $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$ cerute în (1.4):

$$\frac{\partial \mathcal{L}}{\partial(\partial_1 A_2)} = -\frac{1}{4}\,(+4)\,\underbrace{(\partial_1 A_2 - \partial_2 A_1)}_{F_{12}} = -F_{12} = +F_{21} = +F^{21} = -F^{12}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_2 A_1)} = -\frac{1}{4}\,(-4)\,\underbrace{(\partial_1 A_2 - \partial_2 A_1)}_{F_{12}} = +F_{12} = -F_{21} = -F^{21} = +F^{12}$$

şi similar pentru celelalte derivate spaţiu-spaţiu, adică,

$$\frac{\partial \mathcal{L}}{\partial(\partial_1 A_2)} = -F_{12} = -F^{12} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_2 A_3)} = -F_{23} = -F^{23} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_3 A_1)} = -F_{31} = -F^{31}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_2 A_1)} = -F_{21} = -F^{21} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_3 A_2)} = -F_{32} = -F^{32} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_1 A_3)} = -F_{13} = -F^{13}$$

Rezumat: derivatele spatiu-spaţiu sunt: $\Longrightarrow$ $\dfrac{\partial \mathcal{L}}{\partial(\partial_i A_j)} = -F_{ij} = -F^{ij} \quad (i,j = 1,2,3)$

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iar pentru derivatele timp-spaţiu avem

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A_1)} = -\frac{1}{4}\,(-4)\,\underbrace{(\partial_0 A_1 - \partial_1 A_0)}_{F_{01}} = +F_{01} = -F_{10} = +F^{10} = -F^{01}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_1 A_0)} = -\frac{1}{4}\,(+4)\,\underbrace{(\partial_0 A_1 - \partial_1 A_0)}_{F_{01}} = -F_{01} = +F_{10} = -F^{10} = +F^{01}$$

şi similar pentru celelalte derivate timp-spaţiu, adică:

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A_1)} = +F_{01} = -F^{01} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_0 A_2)} = +F_{02} = -F^{02} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_0 A_3)} = +F_{03} = -F^{03}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_1 A_0)} = +F_{10} = -F^{10} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_2 A_0)} = +F_{20} = -F^{20} \quad ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_3 A_0)} = +F_{30} = -F^{30}$$

Rezumat: derivatele timp-spaţiu sunt: $\Longrightarrow$ $\dfrac{\partial \mathcal{L}}{\partial(\partial_0 A_j)} = +F_{0j} = -F^{0j} \quad (j = 1,2,3)$

Deci, $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \quad \Longrightarrow \quad \partial_\mu\!\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial_\mu F^{\mu\nu} \tag{1.6}$$

Inlocuind derivatele (1.5) şi (1.6) în ecuaţia de mişcare Euler-Lagrange
$$\frac{\partial\mathcal{L}}{\partial A_\nu}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)=0 \qquad (1.4)$$

$$\boxed{\partial_\mu F^{\mu\nu}=\mu_0 j^\nu} \qquad (1.7)$$

Am regăsit ecuaţiile Maxwell neomogene.

1.1.1 Ecuaţia de continuitate (conservare curent electromagnetic)

- Am văzut, ecuaţia de continuitate atât în mecanica cuantică nerelativistă cât şi în cea relativistă, este:

$$\partial_\mu j^\mu \equiv \frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{j}=0$$

Folosind ecuaţiie Maxwell neomogene $\partial_\mu F^{\mu\nu}=\mu_0 j^\nu$, ecuaţia de continuitate devine

$$\partial_\mu j^\mu=\partial_\mu\left(\frac{1}{\mu_0}\partial_\nu F^{\nu\mu}\right)=\frac{1}{\mu_0}\partial_\mu\partial_\nu F^{\nu\mu}=0$$

folosind definiţia $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ avem,

$$\partial_\mu F^{\mu\nu}=\partial_\mu\left(\partial^\mu A^\nu-\partial^\nu A^\mu\right)=\partial_\mu\partial^\mu A^\nu-\partial^\nu\left(\partial_\mu A^\mu\right)=\mu_0 j^\mu$$

Folosind calibrarea Lorenz: $\partial_\mu A^\mu=0$, rămâne,

ecuaţia de continuitate pentru potenţialul electromagnetic A^μ

$$\boxed{\partial_\mu\partial^\mu A^\nu=\mu_0 j^\nu} \qquad (1.8)$$

In absenţa curenţilor $j^\nu=0$ obţinem ecuaţia Klein-Gordon pentru particule de masă nulă $m=0$ (fotoni) $\partial_\mu\partial^\mu\psi=0$, adică ecuaţia de propagare a undelor electromagnetice.

A^ν este funcţia de câmp, joacă rolul funcţiei de undă ψ din ecuaţiile cuantice.

1.2 Ecuaţiile de câmp Proca

1.2.1 Lagrangian şi ecuaţiile de câmp Proca

- Lagrangianul Proca, similar cu cel electromagnetic (1.1) însă fără sursă $j^\mu = 0$, dar la care s-a adăugat termenul masic:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu} \qquad (1.9)$$

- Tensorul de câmp Proca $F_{\mu\nu}$ exprimat prin componentele de câmp Proca A_μ , similar cu cele din cazul electromagnetic:

$$\begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \end{cases} \qquad (\mu, \nu = 0, 1, 2, 3)$$

- Exprimarea ecuaţiilor de câmp Proca A_μ, cere determinarea derivatelor Lagrangian-ului din ecuaţiile Euler-Lagrange:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0 \qquad (1.10)$$

- Căutăm întâi derivatele Lagrangian-ului $\dfrac{\partial \mathcal{L}}{\partial A_\nu}$ în ecuaţia Euler-Lagrange (1.10).

In Lagrangian-ul (1.9) avem $\dfrac{m^2}{2}A_\mu A^\mu$ cu produsul scalar

$$A_\mu A^\mu = A_0 A^0 - A_i A^i = A_0 A_0 + A_i A_i,$$

iar derivatele $\dfrac{\partial \mathcal{L}}{\partial A_\nu}$ sunt: $\dfrac{\partial \mathcal{L}}{\partial A_0} = m^2 A_0 = m^2 A^0$; $\dfrac{\partial \mathcal{L}}{\partial A_i} = m^2 A_i = -m^2 A^i$

Atunci I-ul termen al ecuaţiei Euler-Lagrange (1.10) este $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu$ (1.11)

- Acum, derivatele Lagrangian-ului $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$ cerute în (1.10), sunt identice cu cele din cazul câmpului electromagnetic (1.6), vezi pag.7.:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \quad \text{iar în ecuaţia (1.10) avem:} \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = -\partial_\mu F^{\mu\nu} \qquad (1.12)$$

- Inlocuind derivatele (1.11) şi (1.12) în ecuaţia de mişcare Euler-Lagrange (1.10) obţinem Ecuaţiile de câmp Proca:

$$\boxed{\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0} \qquad (1.13)$$
