Strong field electrodynamics

Classical charged particle in strong plane wave

R.A. Ionescu

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Strong field electrodynamics

- charges - electrons (positrons) - fermions — $\mathcal{L}_e$
- EM field — $\mathcal{L}_{EM}$
- interaction (gauge coupling) — $\mathcal{L}_{int}$

Two main problems:

* strong EM field + fermionic vacuum

classical
Grassmann algebra valuated field
(anticommuting numbers)

quantum
Dirac

$\Downarrow$

integrating out the fermionic dof

$\Downarrow$

effective EM Lagrangian
(Euler - Heisenberg)

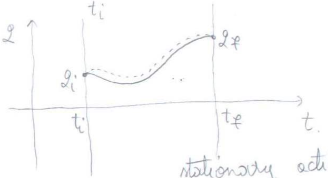
$\longrightarrow$ ** 1 fermion in a given EM field

classical particle description

classical action functional

$$S[q(t)] = \int_{t_i}^{t_f} L(q, \dot{q}; t)\, dt$$

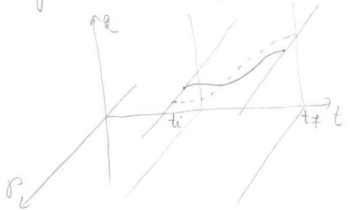
$$q(t_i) = q_i$$
$$q(t_f) = q_f$$



stationary action $\Rightarrow$ Euler-Lagrange eq'n

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

phase space $H(q, p, t)$



$$S[q(t), p(t)] = \int_{t_i}^{t_f} \underbrace{\left(p\dot{q} - H(q, p; t)\right)}_{L(q, \dot{q}(p); t)} dt$$

$\Rightarrow$ Hamilton eq'n

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

QM - probability amplitude q_i at $t_i \to q_f$ at t_f

$$\sum_{\text{all paths } q(t)} e^{\frac{i}{\hbar} S[q(t)]}$$

Feynman

- the sum is dominated by paths close to the classical path (all contributions are almost in phase)

Classical charged particle in a strong EM field

$x_\mu : \underbrace{x_x, x_y, x_z}_{\vec{x}}, t$ $\qquad A_\mu(\vec{x};t) \qquad \vec{A}(\vec{x};t),\ \phi \equiv 0$

$\vec{v} = \frac{d\vec{x}}{dt}$

$$\mathcal{L}_e(\vec{x},\vec{v};t) = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} + e\vec{v}\cdot\vec{A}(\vec{x};t)$$

Legendre transform $\Rightarrow$ Hamiltonian $H_e(\vec{x},\vec{p};t)$

$$p_i = \frac{\partial \mathcal{L}_e}{\partial v_i} = \frac{m v_i}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} + eA_i$$

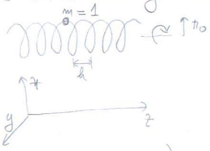
$$\Rightarrow \vec{v}(\vec{p}) = \frac{\vec{p}-e\vec{A}}{\sqrt{m^2+\frac{1}{c^2}(\vec{p}-e\vec{A})^2}}$$

$$H_e(\vec{x},\vec{p};t) = \vec{v}(\vec{p})\cdot\vec{p} - \mathcal{L}_e(\vec{x},\vec{v}(\vec{p});t) =$$

$$= \sqrt{(mc^2)^2 + \left(\vec{p}-e\vec{A}(\vec{x};t)\right)^2 c^2}$$

time dependent Hamiltonian

simple example - particle on a rotating helix



z at t $\Rightarrow x, y$ fixed

$$x = r_0 \cos\left(\frac{z}{h} - \omega t\right)$$

- modulation $r_0(z)$

$$y = r_0 \sin\left(\frac{z}{h} - \omega t\right)$$

$$\frac{dx}{dt} = -\left(\frac{\dot z}{h} - \omega\right) \sin\left(\frac{z}{h} - \omega t\right) r_0(z) + \dot z \, \frac{d r_0(z)}{dz} \cos\left(\frac{z}{h} - \omega t\right)$$

$$\frac{dy}{dt} = \left(\frac{\dot z}{h} - \omega\right) \cos\left(\frac{z}{h} - \omega t\right) r_0(z) + \dot z \, \frac{d r_0(z)}{dz} \sin\left(\frac{z}{h} - \omega t\right)$$

$$L(z, \dot z; t) = T = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \frac{1}{2} r_0^2(z) \left(\frac{\dot z}{h} - \omega\right)^2 + \frac{1}{2}\dot z^2 + \frac{1}{2}\left(\frac{d r_0(z)}{dz}\right)^2 \dot z^2$$

$$p_z = \frac{\partial L}{\partial \dot z} = \frac{r_0^2(z)}{h}\left(\frac{\dot z}{h} - \omega\right) + \left[1 + \left(\frac{d r_0(z)}{dz}\right)^2\right]\dot z$$

$$\dot z = \frac{p_z + \frac{\omega r_0^2(z)}{h}}{1 + \frac{r_0^2(z)}{h^2} + \left(\frac{d r_0(z)}{dz}\right)^2}$$

$$H = p_z \dot z - L = \frac{1}{2}\frac{\left(p_z + \frac{\omega r_0^2(z)}{h}\right)^2}{1 + \frac{r_0^2(z)}{h^2} + \left(\frac{d r_0(z)}{dz}\right)^2} - \frac{1}{2}\omega r_0^2(z)$$

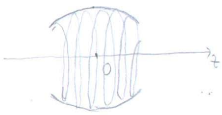
$$r_0(z) = R_0 \quad \Rightarrow H_0 = \frac{1}{2}\frac{\left(p_z + \frac{\omega R_0^2}{h}\right)^2}{1 + \frac{R_0^2}{h^2}} - \frac{1}{2}\omega R_0^2 \Rightarrow \frac{\partial H_0}{\partial z} = 0$$

$$p_z = \text{const} = -\frac{\omega R_0^2}{h} + \Pi_z \quad \Rightarrow H_0 \rightarrow \frac{1}{2}\frac{\Pi_z^2}{1 + \frac{R_0^2}{h^2}}$$

$$m : 1 \longrightarrow 1 + \frac{R_0^2}{h^2}$$ effective mass

$$r_0(z) = R_0 - Bz^2 \quad \Rightarrow \quad r_0^2(z) = R_0^2\left(1 - \frac{B}{R_0}z^2\right)^2 \simeq R_0^2\left(1 - 2\frac{B}{R_0}z^2\right)$$

$$\frac{dr_0}{dz} = -2Bz \quad \Rightarrow \left(\frac{dr_0}{dz}\right)^2 = 4B^2z^2$$



$$H \simeq \frac{1}{2}\,\frac{\Pi_z^2}{1 + \frac{R_0^2}{h^2}} - \frac{1}{2}\,\omega R_0^2\left(1 - 2\frac{B}{R_0}z^2\right)$$

$$\longrightarrow \frac{1}{2}\,\frac{\Pi_z^2}{1 + \frac{R_0^2}{h^2}} + \omega R_0 B z^2 \qquad \rightarrow \text{harmonic oscillator!}$$

$$\frac{d}{dt}\begin{pmatrix} z \\ \Pi_z \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{1 + \frac{R_0^2}{h^2}} \\ -2\omega R_0 B & \end{pmatrix}\begin{pmatrix} z \\ \Pi_z \end{pmatrix}$$

$$H_e(\vec{x},\vec{p};t) = \sqrt{m^2 + (p_x - eA_x)^2 + (p_y - eA_y)^2 + p_z^2} \qquad c = 1$$

$$A_x = a \sin \omega(t-z)$$
$$A_y = -a \cos \omega(t-z)$$

circular polarization

$$H_e = \sqrt{m^2 + e^2a^2 + 2ea\left[p_x \sin\omega(t-z) - p_y \cos\omega(t-z)\right] + p_x^2 + p_y^2 + p_z^2}$$

$$H_e \simeq \sqrt{m^2+e^2a^2} + \frac{ea}{\sqrt{m^2+e^2a^2}}\left[p_x \sin\omega(t-z) - p_y \cos\omega(t-z)\right] + \frac{1}{2}\frac{p_x^2+p_y^2+p_z^2}{\sqrt{m^2+e^2a^2}}$$
$$-\frac{1}{2\sqrt{m^2+e^2a^2}}\frac{e^2a^2}{m^2+e^2a^2}\left[p_x \sin\omega(t-z) - p_y \cos\omega(t-z)\right]^2$$

effective mass

$$p_x = 0 \qquad x_0 = -\frac{1}{\omega}\frac{ea}{\sqrt{m^2+e^2a^2}}\cos\omega t$$

$$p_y = 0 \qquad y_0 = -\frac{1}{\omega}\frac{ea}{\sqrt{m^2+e^2a^2}}\sin\omega t$$

$$p_z = 0 \qquad z_0 = 0$$

relativistic circular motion $\Rightarrow$ synchrotron radiation

$$m_{eff} = m\sqrt{1+\left(\frac{ea}{m}\right)^2}$$

$\eta = \frac{ea}{m}$ (Kirk McDonald)

$\eta \simeq 2\cdot 10^{-6}\sqrt{P}$ diffraction limited focus
$\uparrow$ in W $\quad \lambda \sim 1\mu m$

$1\,TW = 10^{12}\,W$ (E144) $\Rightarrow \eta = 2$

$10\lambda \times 10\lambda \Rightarrow \eta \simeq 0.2$ $\quad m: 511\,keV \rightarrow 530\,keV$

$10\,PW = 10^{16}\,W$ (ELI) $\Rightarrow \eta = 200$

$10\lambda \times 10\lambda \Rightarrow \eta = 20$ $\quad m: 511\,keV \rightarrow 10\,MeV$

E144 $\rightarrow$ (small) shift of the Compton edge ✓

atomic physics $\rightarrow$ multiphoton atom ionization - effective threshold ✓

$$H_e \simeq \sqrt{m^2+e^2e^2} + \frac{ea}{\sqrt{m^2+e^2a^2}}\left(p_x \sin\omega t - p_y\cos\omega t\right) - \frac{1}{2\sqrt{m^2+e^2e^2}}\,\frac{e^2e^2}{m^2+e^2e^2}\left(p_x\sin\omega t - p_y\cos\omega t\right)^2$$

$$+\frac{1}{2}\,\frac{p_x^2+p_y^2+p_z^2}{\sqrt{m^2+e^2e^2}} + \omega\,\frac{ea}{\sqrt{m^2+e^2e^2}}\,z\left(p_x\cos\omega t - p_y\sin\omega t\right)$$

$$x = x_0 + \frac{1}{\sqrt{m^2+e^2e^2}}\,\xi$$

$$y = y_0 + \frac{1}{\sqrt{m^2+e^2e^2}}\,\eta$$

$$z = \frac{1}{\sqrt{m^2+e^2e^2}}\,\zeta$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}\xi\\ \eta\\ \zeta\\ p_x\\ p_y\\ p_z\end{pmatrix} = A\begin{pmatrix}\xi\\ \eta\\ \zeta\\ p_x\\ p_y\\ p_z\end{pmatrix}$$

$$A = \left(\begin{array}{ccc|ccc}
0 & 0 & \frac{\omega}{a}\frac{e^2e^2}{m^2+e^2e^2}\cos\omega t & 1-\frac{e^2e^2}{m^2+e^2e^2}\sin^2\omega t & \frac{e^2e^2}{m^2+e^2e^2}\sin\omega t\cos\omega t & 0\\
0 & 0 & -\frac{\omega}{a}\frac{e^2e^2}{m^2+e^2e^2}\sin\omega t & \frac{e^2e^2}{m^2+e^2e^2}\sin\omega t\cos\omega t & 1-\frac{e^2e^2}{m^2+e^2e^2}\cos^2\omega t & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
\hline
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\omega\frac{ea}{\sqrt{m^2+e^2e^2}}\cos\omega t & \omega\frac{ea}{\sqrt{m^2+e^2e^2}}\sin\omega t & 0
\end{array}\right)$$

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \\ p_x \\ p_y \\ p_t \end{pmatrix}(t) = Z(t) \begin{pmatrix} \xi \\ \eta \\ \zeta \\ p_x \\ p_y \\ p_t \end{pmatrix}(t=0)$$ where

$$\frac{d Z(t)}{dt} = A(t) Z(t) \quad , \; Z(t=0) = \mathbb{1}$$

matrix homogeneous linear differential eq'n with periodic coefficients (period $T = \frac{1}{2\pi\omega}$)

Floquet th $\quad Z(t) = \underset{\substack{\uparrow \\ \text{periodic} \\ \text{matrix}}}{e^{\Lambda(t)}} \; \underset{\substack{\uparrow \\ \text{constant matrix}}}{e^{tF}}$

recursive Magnus expansion (F. Casas et al. - JPA 34 (2001) 3379)

$$\Lambda(t) = \sum_{n=1}^{\infty} \Lambda_n(t)$$

$$F = \sum_{n=1}^{\infty} F_n$$

$$F_1 = \frac{1}{T}\int_0^T A(\tau)d\tau \qquad F_2 = \frac{1}{2T}\int_0^T [A(\tau)+F_1, \Lambda_1(\tau)]d\tau$$

$$\Lambda_1(t) = \int_0^t A(\tau)d\tau - tF_1 \qquad \Lambda_2(t) = \frac{1}{2}\int_0^t [A(\tau)+F_1, \Lambda_1(\tau)]d\tau - tF_2$$

Conclusions:

- electron/positron in plane wave → effective mass ⇒ change in Compton edge
- there is a system of reference (in which the mean velocity is zero) in which the motion is circular ..
- increased effective mass ⇒ suppressing pair production !

- Multumesc!