Procese QED în câmpuri laser intense
tema 37

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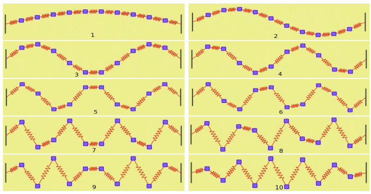
”Verbum sapienti sat est”

Cuprins

Partea I

De la Mecanica Clasică (particule) la Câmpuri Clasice (unde)

Lagrangian sisteme oscilante Ecuaţia Euler-Lagrange în mecanică: $\frac{\partial L}{\partial y_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right)=0 \qquad (i=1,...,n)$	Ecuaţia Euler-Lagrange pentru câmp $\frac{\partial\mathcal{L}}{\partial\varphi}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}-\nabla\frac{\partial\mathcal{L}}{\partial(\nabla\varphi)}=0$ sau $\frac{\partial\mathcal{L}}{\partial\varphi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)=0$	Trecere mecanică clasică ⟶ câmpuri clasice $y\to\varphi(x,t)$; $p\to\pi(x,t)$ $L(x,\dot{x})\to\mathcal{L}(\varphi,\dot{\varphi})$; $\pi(x,t)=\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}$ $H(x,p)\to\mathcal{H}(\varphi,\pi)$; $\mathcal{H}=\pi\dot{\varphi}-\mathcal{L}$
Lagrangian oscilator armonic $L(y,\dot{y})=T-V=\frac{m\dot{y}^2}{2}-\frac{ky^2}{2}$	Ecuaţia Euler-Lagrange $-ky=m\ddot{y}$	Soluţia, ecuaţia de evoluţie oscilator armonic: $y(t)=y_0\left(e^{i\omega t}+e^{-i\omega t}\right)$; unde $\omega=\sqrt{k/m}$
Sistem cuplat de n oscilatori transversali de masă m $L(y_i,\dot{y}_i)=T-V=\sum_{i=1}^{n}\frac{m\dot{y}_i^2}{2}-\frac{k_t}{2}\sum_{i=1}^{n+1}\left(y_i-y_{i-1}\right)^2$ unde $k_t=\tau/\Delta x$ 	Sistem ecuaţii Euler-Lagrange prin coordonatele de mod normal $Y_i(t)=y_i(t)-y_i^e$ $\begin{cases}-k_t\left(2Y_1-Y_2\right)=m\ddot{Y}_1\\-k_t\left(-Y_1+2Y_2-Y_3\right)=m\ddot{Y}_2\\-k_t\left(-Y_2+2Y_3-Y_4\right)=m\ddot{Y}_3\\\ldots\ldots\ldots\\-k_t\left(-Y_{n-2}+2Y_{n-1}-Y_n\right)=m\ddot{Y}_{n-1}\\-k_t\left(-Y_{n-1}+2Y_n\right)=m\ddot{Y}_n\end{cases}$	Soluţiile de mod normal de oscilaţie armonică: $Y_i(t)=\underbrace{A\sin\kappa_j x_i}\cos\omega_j t$ i indică poziţia masei oscilante din lanţ (longitudinală) j indică modul normal de oscilaţie (transversală). Frecvenţele ω_j de mod normal de oscilaţie ale lanţului (grade de libertate, cuantificate) $\omega_j=2\,\omega_0\sin\left(\frac{j\pi}{2(n+1)}\right) \qquad (j=1,2,\ldots,n)$ unde $\omega_0=\sqrt{\frac{k_t}{m}}$
Coardă oscilantă $\mathcal{L}=\overbrace{\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2}^{T}-\overbrace{\frac{\rho v^2}{2}\left(\frac{\partial y}{\partial x}\right)^2}^{V}$ unde $v^2=\frac{\tau}{\rho}$	Ecuaţia de propagare a undelor $\frac{\partial^2 y}{\partial t^2}-v^2\frac{\partial^2 y}{\partial x^2}=0$	Soluţia de unde staţionare pe domeniul $x=0\div L$ $y_n(x,t)=A_n(t)\sin\left(\frac{n\pi}{L}x\right)$
Lagrangian de câmp scalar liber $\mathcal{L}=\overbrace{\frac{1}{2}\frac{1}{c^2}\left(\partial_t\varphi\right)^2}^{T}-\overbrace{\frac{1}{2}\left(\nabla\varphi\right)^2}^{V}$ $\mathcal{L}=\frac{1}{2}\left(\partial_\mu\varphi\right)^2$	Ecuaţia de evoluţie a câmpului scalar (ecuaţia de propagare a undelor) $\left(\frac{1}{c^2}\partial_t^2-\nabla^2\right)\varphi=0$ sau $(\partial_\mu\partial^\mu)\,\varphi=0$	Soluţia de undă plană în mişcare $\varphi(\vec{x},t)=A(\vec{k})\,e^{-i(\omega t-\vec{k}\cdot\vec{x})}+B(\vec{k})\,e^{i(\omega t-\vec{k}\cdot\vec{x})}$ sau $\varphi(x)=A(\vec{k})\,e^{-ik\cdot x}+B(\vec{k})\,e^{ik\cdot x}$
Lagrangean de Câmp Scalar Real Masiv - Klein-Gordon (cu $\hbar=c=1$) $\mathcal{L}=\frac{1}{2}\left[(\partial_t\varphi)^2-(\nabla\varphi)^2-m^2\varphi^2\right]$ $\equiv\frac{1}{2}\left[(\partial_\mu\varphi)^2-m^2\varphi^2\right]$	Ecuaţia de evoluţie a câmpului Klein-Gordon $(\partial_t^2-\nabla^2+m^2)\varphi=0$ sau $(\partial_\mu\partial^\mu+m^2)\varphi=0$	Soluţia de undă plană în mişcare $\varphi(\vec{x},t)=A(\vec{k})\,e^{-i(\omega t-\vec{k}\cdot\vec{x})}+B(\vec{k})\,e^{i(\omega t-\vec{k}\cdot\vec{x})}$ sau $\varphi(x)=A(\vec{k})\,e^{-ik\cdot x}+B(\vec{k})\,e^{ik\cdot x}$
Lagrangean de Câmp Schrödinger $\mathcal{L}=i\hbar\,\varphi^*\dot{\varphi}-\frac{\hbar^2}{2m}\nabla\varphi^*\cdot\nabla\varphi-V\varphi^*\varphi$	Ecuaţia de evoluţie a câmpului Schrödinger $i\hbar\frac{\partial\varphi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\varphi+V\,\varphi$ $-i\hbar\frac{\partial\varphi^*}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\varphi^*+V\varphi^*$	Soluţia de undă plană în mişcare $\varphi(\vec{x},t)=\alpha\,e^{-i(\omega t-\vec{k}\cdot\vec{x})}$ $\varphi^*(\vec{x},t)=\alpha^*\,e^{i(\omega t-\vec{k}\cdot\vec{x})}$ absentă în Mec.Cuantică
Lagrangean de Câmp Dirac $\mathcal{L}=i\,\hbar\,\bar{\varphi}\,\gamma^\mu\partial_\mu\varphi-mc\,\bar{\varphi}\,\varphi$ unde $\bar{\varphi}=\varphi^\dagger\gamma^0$	Ecuaţia de evoluţie a câmpurilor Dirac $i\hbar\,\gamma^\mu\partial_\mu\varphi-mc\varphi=0$ $i\hbar\,\partial_\mu\bar{\varphi}\gamma^\mu+mc\bar{\varphi}=0$	Soluţiile de undă plană Dirac Soluţia de $E>0$ $\varphi(\vec{x},t)=u(\vec{p})\,e^{-i\,p\cdot x}$ Soluţia de $E<0$ $\bar{\varphi}(\vec{x},t)=v(\vec{p})\,e^{+i\,p\cdot x}$
Lagrangean de Câmp Maxwell $\mathcal{L}=\frac{1}{2}\left(\epsilon_0 E^2-\frac{1}{\mu_0}B^2\right)-\rho\varphi+\vec{J}\cdot\vec{A}$ sau $\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\mu_0\,j_\mu A^\mu$ unde $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$; $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$	Ecuaţiile de câmp Maxwell $\begin{cases}\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}\\\vec{\nabla}\times\vec{B}-\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{J}\end{cases}$ sau $\partial_\mu\partial^\mu A^\nu=\mu_0 j^\nu$	Soluţie de undă plană Maxwell $A^\mu=\epsilon^\mu(\vec{q})\,e^{-i\,q\cdot x}$

Partea II

De la Mecanica Cuantică (unde) la Câmpuri Cuantice (particule)

Cuantificarea I (particule → unde)

Asocierea corpuscul - undă: $p=\dfrac{h}{\lambda}=\hbar k \quad ; \quad E=h\nu=\hbar\omega$

Operatori pt. mărimi fizice: $x\to\hat{x} \ ; \ p\to\hat{p}=-i\hbar\partial_x \ ; \ E\to\hat{E}=i\hbar\partial_t$

Relaţie de comutare operatori: $[\hat{x},\hat{p}]=i\hbar$

Cuantificarea a II-a (unde → particule)

Operatori de câmp: $\varphi\to\hat{\varphi} \quad ; \quad \pi\to\hat{\pi}$

$\varphi^*\to\hat{\varphi}^\dagger \quad ; \quad \pi^*\to\hat{\pi}^\dagger$

Ec. evoluţie operatori: $\boxed{i\hbar\dot{\hat{\varphi}}=\left[\hat{\varphi},\hat{\mathcal{H}}\right]}$ (Ec. Heisenberg)

unde $\hat{\mathcal{H}}(\hat{\varphi}^i,\hat{\pi}_i)=\hat{\pi}_i\dot{\hat{\varphi}}^i-\mathcal{L}$

Cuantificare oscilator armonic - operatori de creare $\hat{a}^\dagger$ şi anihilare $\hat{a}$ stări

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{m\omega^2}{2}\hat{x}^2=\left(\hat{X}^2+\hat{P}^2\right)\frac{\hbar\omega}{2}=\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\frac{\hbar\omega}{2}$$

$$\hat{X}=\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} \ ; \ \hat{P}=\sqrt{\frac{1}{m\omega\hbar}}\,\hat{p} \quad \Big| \quad \hat{a}=\frac{1}{\sqrt{2}}\left(\hat{X}+i\hat{P}\right) \ ; \ \hat{a}^\dagger=\frac{1}{\sqrt{2}}\left(\hat{X}-i\hat{P}\right)$$

Comutatori: $[\hat{x},\hat{p}]=i\hbar \quad ; \quad \left[\hat{X},\hat{P}\right]=i \quad ; \quad [\hat{a},\hat{a}^\dagger]=1$

$$\begin{cases}\hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle\\ \hat{a}^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle\end{cases} \quad \Big| \quad \begin{cases}\hat{N}|n\rangle=\hat{a}^\dagger\hat{a}|n\rangle=\\ =\sqrt{n}\sqrt{n}\,|n\rangle=n\,|n\rangle\end{cases}$$

cu $\hat{a}|0\rangle=0$

$$\hat{H}\,|n\rangle=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)|n\rangle=\hbar\omega\left(n+\frac{1}{2}\right)|n\rangle=E_n\,|n\rangle$$

$\boxed{E_n=\left(n+\frac{1}{2}\right)\hbar\omega \quad (n=0,1,2,\dots)}$

Soluţia generală ca superpoziţie de unde plane discrete

$$\hat{\varphi}(x)=\sum_{\vec{k}}\frac{1}{\sqrt{V\,2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}\,e^{-ik\cdot x}+\hat{b}^\dagger_{\vec{k}}\,e^{ik\cdot x}\right) \ ; \hat{\pi}=\dot{\hat{\varphi}}$$

Soluţia generală ca superpoziţie de unde plane continui

$$\hat{\varphi}(x)=\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}\,e^{-ik\cdot x}+\hat{b}^\dagger_{\vec{k}}\,e^{ik\cdot x}\right) \ ; \hat{\pi}=\dot{\hat{\varphi}}$$

Relaţii de comutare pt. operatori de câmp, la acelaşi timp

$$\begin{cases}\left[\hat{\varphi}^i(\vec{x},t),\hat{\pi}_j(\vec{y},t)\right]=i\,\delta^i_j\,\delta(\vec{x}-\vec{y})\\ \left[\hat{\varphi}^i,\hat{\varphi}^j\right]=\left[\hat{\pi}_i,\hat{\pi}_j\right]=0\end{cases}$$

$$\begin{cases}\left[\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{k}'}\right]=\left[\hat{b}_{\vec{k}},\hat{b}^\dagger_{\vec{k}'}\right]=\delta_{\vec{k}\vec{k}'} \ \text{(discret)}\\ \qquad\qquad\qquad\qquad\ =\delta(\vec{k}-\vec{k}') \ \text{(continuu)}\end{cases}$$

Ecuaţia Klein-Gordon ($\hbar=c=1$)

$$(\partial_t^2-\nabla^2+m^2)\,\varphi=0 \qquad \text{sau} \qquad (\partial_\mu\partial^\mu+m^2)\,\varphi=0$$

Soluţia generală ca superpoziţie de unde plane discrete (pe volum V finit)

$$\varphi(t,\vec{x})=\sum_{\vec{k}}\frac{1}{\sqrt{2\omega_k V}}\Big(\underbrace{a_{\vec{k}}\,e^{-i(\omega t-\vec{k}\cdot\vec{x})}}_{\varphi_+}+\underbrace{a^*_{\vec{k}}\,e^{i(\omega t-\vec{k}\cdot\vec{x})}}_{\varphi_-}\Big)$$

$$\hat{E}\varphi_+\equiv\frac{\partial\varphi_+}{\partial t}=i\hbar(-i\omega_k)\frac{a_{\vec{k}}\,e^{-i(\omega_k t-kx)}}{\sqrt{2\omega_k V}}=\underbrace{\hbar\omega_k}_{E_+}\varphi_+ \quad \boxed{E_+=\hbar\omega_k>0}$$

$$\hat{E}\varphi_-\equiv i\hbar\frac{\partial\varphi_-}{\partial t}=i\hbar(i\omega_k)\frac{a^*_{\vec{k}}\,e^{i(\omega_k t-kx)}}{\sqrt{2\omega_k V}}=\underbrace{-\hbar\omega_k}_{E_-}\varphi_- \quad \boxed{E_-=-\hbar\omega_k<0}$$

Soluţia generală ca superpoziţie de unde plane continui, prin $\sum_{\vec{k}}\to\int d^3\vec{k}$

$$\varphi(x)=\int d^4k\left(\alpha_{\vec{k}}\,e^{-i\,k\cdot x}+\alpha^*_{\vec{k}}\,e^{i\,k\cdot x}\right)$$

$$=\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_k}}\left(\alpha_{\vec{k}}\,e^{-i(\omega t-\vec{k}\cdot\vec{x})}+\alpha^*_{\vec{k}}\,e^{i(\omega t-\vec{k}\cdot\vec{x})}\right)$$

aceasta este o dezvoltare Fourier.

Cuantificare câmpuri Klein-Gordon

$$\hat{H}_{KG}=\int\underbrace{\frac{1}{2}\left(\hat{\pi}^2+\left(\hat{\nabla}\hat{\varphi}\right)^2+m^2\hat{\varphi}^2\right)}_{\hat{\mathcal{H}}_{KG}}d^3\vec{x} \quad ; \quad \pi(\vec{x})=\dot{\hat{\varphi}}(\vec{x})$$

Ecuaţiile de evoluţie operatori de câmp Klein-Gordon $\hat{\varphi}$ şi $\hat{\pi}$

$$\begin{cases}i\,\hbar\dfrac{\partial\hat{\varphi}}{\partial t}=\left[\hat{\varphi},\hat{H}_{KG}\right]=i\hbar\,\hat{\pi}(\vec{x})\\ i\,\hbar\dfrac{\partial\hat{\pi}}{\partial t}=\left[\hat{\pi},\hat{H}_{KG}\right]=i\hbar\left(\hat{\nabla}^2-m^2\right)\hat{\varphi}\end{cases}$$

Soluţiile operator de câmp $\hat{\varphi}$ şi $\hat{\pi}$ la acelaşi timp sunt:

$$\begin{cases}\hat{\varphi}(\vec{x})=\displaystyle\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_k}}\left(\hat{a}_k\,e^{i\,\vec{k}\cdot\vec{x}}+\hat{a}^\dagger_k\,e^{-i\,\vec{k}\cdot\vec{x}}\right)\\ \hat{\pi}(\vec{x})=\displaystyle\int\frac{d^3\vec{k}}{\sqrt{(2\pi)^3\,2\omega_k}}(-i\omega_k)\left(\hat{a}_k\,e^{i\,\vec{k}\cdot\vec{x}}-\hat{a}^\dagger_k\,e^{-i\,\vec{k}\cdot\vec{x}}\right)\end{cases}$$

Relaţii comutare operatori de câmp KG la acelaşi timp

$$\left[\hat{\varphi}(\vec{x},t_0),\hat{\pi}'(\vec{x}',t_0)\right]=i\,\hbar\,\delta^3(\vec{x}-\vec{x}')$$

Cu trecerea: $(\hat{\varphi},\hat{\pi})\to(\hat{a}_k,\hat{a}^\dagger_k)$

$$\begin{cases}\hat{\alpha}_k=\dfrac{1}{2}\displaystyle\int\frac{d^3\vec{x}}{(2\pi)^3}\left(\hat{\varphi}(\vec{x})+\frac{i}{\omega_k}\dot{\hat{\varphi}}(\vec{x})\right)e^{-i\,\vec{k}\cdot\vec{x}}\\ \hat{\alpha}^\dagger_k=\dfrac{1}{2}\displaystyle\int\frac{d^3\vec{x}}{(2\pi)^3}\left(\hat{\varphi}(\vec{x})-\frac{i}{\omega_k}\dot{\hat{\varphi}}(\vec{x})\right)e^{i\,\vec{k}\cdot\vec{x}}\end{cases}$$

Relaţiile de comutare: $\left[\hat{a}_k,\hat{a}^\dagger_{k'}\right]=\delta^{(3)}\left(\vec{k}-\vec{k}'\right)$

$$\hat{\mathcal{H}}_{KG}=\frac{1}{2}\int d^3\vec{k}\ \omega_k\left(\hat{a}_k\hat{a}^\dagger_k+\hat{a}^\dagger_k\hat{a}_k\right)=\int d^3\vec{k}\ \omega_k\left(\hat{a}^\dagger_k\hat{a}_k+\frac{1}{2}\right)$$

Operatorul densitate de energie pt. mode k osc. câmp KG (cu $\hbar$): $\hat{E}_k=\hbar\omega_k\left(\hat{a}^\dagger_k\hat{a}_k+\frac{1}{2}\right)$

Ecuaţia Dirac

$$\left(\frac{i}{c}\gamma^0\partial_t+i\gamma\cdot\nabla-\frac{mc}{\hbar}\right)\varphi=0 \quad \text{sau} \quad \left(i\gamma^\mu\partial_\mu-\frac{mc}{\hbar}\right)\varphi=0$$

Soluţia generală ca superpoziţie de unde plane

$$\varphi(x)=\underbrace{u^{(1,2)}(\vec{p})\,e^{-i\,p\cdot x/\hbar}}_{\varphi_+}+\underbrace{v^{(2,1)}(\vec{p})\,e^{i\,p\cdot x/\hbar}}_{\varphi_-}$$

$$(\gamma^\mu p_\mu-mc)\,u(\vec{p})=0 \quad ; \quad (\gamma^\mu p_\mu+mc)\,v(\vec{p})=0$$

$$\begin{cases}u_A^{(1,2)}=\chi^{(1,2)} \quad ; \quad u_B^{(1,2)}=\frac{\vec{\sigma}\cdot\vec{p}c}{E+mc^2}\,\chi^{(1,2)}\\ v_A^{(1,2)}=\frac{-\vec{\sigma}\cdot\vec{p}c}{|E|+mc^2}\,\chi^{(1,2)} \quad ; \quad v_B^{(1,2)}=\chi^{(1,2)}\end{cases}$$

Ecuaţia de câmp electromagnetic

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{A}=0 \quad ; \quad \vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t} \quad ; \quad \vec{B}=\vec{\nabla}\times\vec{A}$$

Soluţia generală: $\vec{A}(\vec{r},t)=\vec{\epsilon}\sqrt{\dfrac{\hbar}{2\epsilon_0\omega V}}\left(a\,e^{-i(\omega t-\vec{k}\cdot\vec{r})}+a^*e^{i(\omega t-\vec{k}\cdot\vec{r})}\right)$

Cuantificare câmp electromagnetic

Soluţii operator de câmp EM $\hat{A}(\vec{r},t)$

$$\hat{\vec{A}}(\vec{r},t)=\sum_{k,\lambda}\vec{\epsilon}_\lambda\sqrt{\frac{\hbar}{\epsilon_0 2\omega_k V}}\left(\hat{a}_{k\lambda}e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}+\hat{a}^\dagger_{k\lambda}e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\right)$$

Hamiltonian câmp EM: $\hat{H}_{EM}=\dfrac{1}{2}\displaystyle\int_V\left(\epsilon_0\hat{E}^2+\frac{\hat{B}^2}{\mu_0}\right)d^3r$

$$\hat{H}_{EM}=\frac{1}{2}\sum_{k,\lambda}\hbar\omega_k\left(\hat{a}^\dagger_{k,\lambda}\hat{a}_{k,\lambda}+\hat{a}_{k,\lambda}\hat{a}^\dagger_{k,\lambda}\right)=\sum_{k,\lambda}\hbar\omega_k\Big(\underbrace{\hat{a}^\dagger_{k,\lambda}\hat{a}_{k,\lambda}}_{\hat{N}_{k\lambda}}+\frac{1}{2}\Big)$$

Operatorul densitate de energie pt. mode k oscilaţie câmp EM: $\hat{E}_k=\hbar\omega_k\left(\hat{N}_{k\lambda}+\frac{1}{2}\right)$