Procese QED în câmpuri laser intense
tema 39

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March 11, 2022

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”Verbum sapienti sat est”

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1.1 Cuantificare câmp Dirac

- Cuantificarea oricăror câmpuri se face prin exprimarea amplitudinilor din dezvoltarea generală Fourier a câmpului, ca superpoziţie de câmpuri libere.

  Dacă în teoria câmpurilor clasice, amplitudinile dezvoltării după câmpuri libere se determină ca *transformata Fourier* a acestor dezvoltări, în teoria câmpurilor cuantice, amplitudinile dezvoltării după câmpurile cuantice libere, ca vectori de bază ai unui spaţiu Hilbert, se determină prin proiectarea pe fiecare din aceste câmpuri cuantice libere.

  Să exprimăm întâi Lagrangian-ul şi Hamiltonian-ul de câmp Dirac cuantic.

- Lagrangian-ul de *câmpuri cuantice Dirac* $\hat{\varphi}$ şi $\hat{\overline{\varphi}}$:

$$\boxed{\mathcal{L} = i\hbar\hat{\overline{\varphi}}\gamma^\mu\partial_\mu\hat{\varphi} - mc\hat{\overline{\varphi}}\hat{\varphi}} \tag{1.1}$$

- Câmpul Dirac de impuls conjugat canonic, este dat de operatorul:

$$\hat{\pi} = \frac{\partial\mathcal{L}}{\partial\dot{\hat{\varphi}}} = i\hbar\hat{\overline{\varphi}}\gamma^0 = i\hbar\hat{\varphi}^\dagger \tag{1.2}$$

- Folosind ecuaţia Dirac $\boxed{i\hbar\,\gamma^\mu\partial_\mu\hat{\varphi} - mc\,\hat{\varphi} = 0}$ [1]

  Densitatea de Hamiltonian de *câmp Dirac* cuantic,

$$\mathcal{H}_D = \hat{\pi}\dot{\hat{\varphi}} - \mathcal{L} = \underbrace{\overbrace{i\hbar\hat{\overline{\varphi}}\gamma^0}^{\hat{\pi}}\,\partial_0\hat{\varphi} - i\hbar\hat{\overline{\varphi}}\gamma^\mu\partial_\mu\hat{\varphi}}_{-i\hbar\hat{\overline{\varphi}}\gamma^i\partial_i\hat{\varphi}} + mc\,\hat{\overline{\varphi}}\hat{\varphi} = \hat{\overline{\varphi}}\,\underbrace{\left(-i\hbar\,\gamma^i\partial_i + mc\right)}_{-i\hbar\gamma^0\partial_0 \text{ din ec.Dirac}}\,\hat{\varphi} \tag{1.3}$$
$$= \hat{\overline{\varphi}}\left(i\hbar\gamma^0\partial_0\hat{\varphi}\right) = i\hbar\,\hat{\varphi}^\dagger\partial_0\hat{\varphi}$$

- Hamiltonian-ul de câmp Dirac cuantic, folosind (1.3), este:

$$\hat{H}_D = \int \mathcal{H}_D\, d^3\vec{x} \tag{1.4}$$

  deoarece $i\hbar\frac{d\hat{H}}{dt} = \left[\hat{H}, \hat{H}\right] \equiv 0 \Rightarrow \hat{H}_D$ e independent de t

- Ecuaţia Heisenberg de mişcare pentru operatorul de câmp Dirac cuantic $\hat{\varphi}(\vec{x}, t)$ este:

$$i\,\hbar\frac{\partial\hat{\varphi}}{\partial t} = \left[\hat{\varphi}, \hat{H}_D\right] \tag{1.5}$$

- Soluţiile de undă plană de câmp liber Dirac $\hat{\varphi}(\vec{x}, t)$ şi $\hat{\overline{\varphi}}(\vec{x}, t)$, se folosesc acum ca operatori de câmpuri cuantice $\hat{\varphi}$ şi $\hat{\overline{\varphi}}$, exprimate cu ajutorul spinorilor Dirac $u_s(\vec{p})$ şi $v_s(\vec{p})$, sunt:

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[1] Atenţie ! γ^μ nu este 4-vector, iar ∂_μ este 4-vector covariant cu metrica $(+, +, +, +)$.
Apoi, deşi trecerea la câmpuri cuantice se face prin $\dot{q} \to \partial_\mu\hat{\varphi}$, pentru densităţi de Hamiltonian şi de câmpuri se împarte cu $d^3\vec{x}$, atunci rămâne valabilă trecerea $\dot{q} \to \partial_0\hat{\varphi}$.

$$\hat{\varphi}(\vec{x},t) = u_s(\vec{p})\, e^{-i\,p\cdot x/\hbar} \qquad \hat{\overline{\varphi}}(\vec{x},t) = v_s(\vec{p})\, e^{+i\,p\cdot x/\hbar} \tag{1.6}$$

$$u_{(1,2)} = \begin{pmatrix} 1 \\ 0 \\ \frac{p_3 c}{E+mc^2} \\ \frac{(p_1+ip_2)c}{E+mc^2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{(p_1-ip_2)c}{E+mc^2} \\ \frac{-p_3 c}{E+mc^2} \end{pmatrix} \quad ; \quad v_{(1,2)} = \begin{pmatrix} \frac{p_3 c}{|E|+mc^2} \\ \frac{(p_1+ip_2)c}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{(p_1-ip_2)c}{|E|+mc^2} \\ \frac{-p_3 c}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

- Soluţia generală de câmp Dirac o scriem ca o superpoziţie de unde plane, de data asta ca operatori luaţi <u>la acelaşi timp</u>, ca în cazul oricărui câmp cuantic,

$$\begin{cases} \hat{\varphi}(\vec{x},t) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega} \Big(\hat{a}_s(\vec{p})\, u_s(\vec{p})\, e^{-i\,p\,\cdot\,x} + \hat{b}_s^\dagger(\vec{p})\, v_s(\vec{p})\, e^{i\,p\,\cdot\,x} \Big) \\ \hat{\varphi}^\dagger(\vec{x},t) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega} \Big(\hat{b}_s(\vec{p})\, v_s^\dagger(\vec{p})\, e^{-i\,p\,\cdot\,x} + \hat{a}_s^\dagger(\vec{p})\, u_s^\dagger(\vec{p})\, e^{i\,p\,\cdot\,x} \Big) \\ \hat{\overline{\varphi}}(\vec{x},t) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega} \Big(\hat{b}_s(\vec{p})\, \overline{v}_s(\vec{p})\, e^{-i\,p\,\cdot\,x} + \hat{a}_s^\dagger(\vec{p})\, \overline{u}_s(\vec{p})\, e^{i\,p\,\cdot\,x} \Big) \end{cases} \tag{1.7}$$

- Coeficienţii $\hat{a}_s(\vec{p})$ şi $\hat{b}_s(\vec{p})$ au structură de operatori, pe când $u_s(\vec{p})$ şi $v_s(\vec{p})$ sunt cunoscuţii spinori Dirac.

- Va trebui să ţinem cont de relaţiile de ortogonalitate pentru spinori:

$$\overline{u}_r(\vec{p}) u_s(\vec{p}) = \delta_{rs} \qquad \overline{v}_r(\vec{p}) u_s(\vec{p}) = 0$$
$$\overline{v}_r(\vec{p}) v_s(\vec{p}) = -\delta_{rs} \qquad \overline{u}_r(\vec{p}) v_s(\vec{p}) = 0 \tag{1.8}$$

- De asemenea, avem nevoie de identităţile:

$$\overline{u}_r(\vec{p})\, \gamma^0 u_s(\vec{p}) = u_r^\dagger(\vec{p})\, u_s(\vec{p}) = \frac{\omega}{m}\delta_{rs} \tag{1.9}$$

$$\overline{v}_r(\vec{p})\, \gamma^0 v_s(\vec{p}) = v_r^\dagger(\vec{p})\, v_s(\vec{p}) = \frac{\omega}{m}\delta_{rs} \tag{1.10}$$

Aceste identităţi pot fi demonstrate după cum urmează:

Pentru I-a identitate (1.9), plecăm de la: $\boxed{u_r^\dagger(\vec{p})\, u_s(\vec{p}) = \overline{u}_r(\vec{p})\, \gamma^0 u_s(\vec{p})}$

Folosind ecuaţiile Dirac pentru spinori:

$$\begin{array}{ccc} (\gamma^\mu p_\mu - m)\, u_s(\vec{p}) = 0 & \longrightarrow & \overline{u}_s(\vec{p})\, (\gamma^\mu p_\mu - m) = 0 \\ \downarrow & & \downarrow \\ u_s(\vec{p}) = \frac{1}{m}\gamma^\mu p_\mu u_s(\vec{p}) & & \overline{u}_s(\vec{p}) = \frac{1}{m}\overline{u}_s(\vec{p})\, \gamma^\mu p_\mu \end{array}$$

$$
\begin{aligned}
u_r^\dagger(\vec{p})u_s(\vec{p}) &= \frac{1}{2}\Big(\overbrace{\overline{u}_r(\vec{p})\gamma^0}^{u_r^\dagger}\underbrace{u_s(\vec{p})} + \overbrace{\underbrace{\overline{u}_r(\vec{p})}\gamma^0}^{u_r^\dagger}u_s(\vec{p})\Big) \\
&= \frac{1}{2m}\Big(\overbrace{\overline{u}_r(\vec{p})\gamma^0}^{u_r^\dagger}\overbrace{\gamma^\mu p_\mu u_s(\vec{p})}^{u_s(\vec{p})} + \overbrace{\overline{u}_r(\vec{p})\gamma^\mu p_\mu}^{\overline{u}_r(\vec{p})}\gamma^0 u_s(\vec{p})\Big) \\
&= \frac{1}{2m}\overline{u}_r(\vec{p})\underbrace{\{\gamma^0,\gamma^\mu\}}_{2g^{0\mu}}p_\mu u_s(\vec{p})
\end{aligned}
$$

$\{\gamma^0,\gamma^\mu\} = 2g^{0\mu}$ vezi <u>Anticomutativitate matrici γ^μ</u>

$$
u_r^\dagger(\vec{p})u_s(\vec{p}) = \frac{\omega}{2m}\overline{u}_r(\vec{p})u_s(\vec{p}) = \frac{\omega}{m}\delta_{rs}
$$

1.1.1 Amplitudinile Fourier din dezvoltarea de câmp Dirac

Determinarea amplitudinilor $\hat{a}_s(\vec{p})$

- Dezvoltarea Fourier (1.7) a câmpului $\hat{\varphi}(\vec{x},t)$, este:

$$
\hat{\varphi}(\vec{x},t) = \sum_s\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\Big[\hat{a}_s(\vec{p})u_s(\vec{p})e^{-i\,p\cdot x} + \hat{b}_s^\dagger(\vec{p})v_s(\vec{p})e^{i\,p\cdot x}\Big] \qquad (1.11)
$$

Proiectăm (1.11) pe componenta armonică $e^{i\,p'\cdot x}$ de funcţii ortogonale:

$$
\begin{aligned}
\boxed{\int d^3\vec{x}\,\hat{\varphi}(\vec{x},t)e^{i\,p'\cdot x}} &= \sum_s\int d^3\vec{x}\frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\Big[\hat{a}_s(\vec{p})u_s(\vec{p})e^{-i(p-p')\cdot x} + \hat{b}_s^\dagger(\vec{p})v_s(\vec{p})e^{i\,(p+p')\cdot x}\Big] \\
&= \sum_s\int d^3\vec{x}\frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\Big[\hat{a}_s(\vec{p})u_s(\vec{p})\,e^{-i\,(\omega-\omega')t}\,e^{i\,(\vec{p}-\vec{p}')\cdot\vec{x}} + \\
&\qquad + \hat{b}_s^\dagger(\vec{p})v_s(\vec{p})\,e^{i\,(\omega'+\omega)t}\,e^{-i\,(\vec{k}'+\vec{k})\cdot\vec{x}}\Big]
\end{aligned}
$$

Pentru integrala după x, folosim funcţia δ:

$$
\int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}-\vec{k}')\cdot\vec{x}} = \delta^3(\vec{k}-\vec{k}') \quad ; \quad \int\frac{d^3\vec{x}}{(2\pi)^3}e^{i\,(\vec{k}+\vec{k}')\cdot\vec{x}} = \delta^3(\vec{k}'+\vec{k}) \qquad (1.12)
$$

Atunci, integrala de mai sus după $d^3\vec{x}$ va fi:

$$
\begin{aligned}
\boxed{\int d^3\vec{x}\,\hat{\varphi}(\vec{x},t)\,e^{i\,p'\cdot x}} &= \sum_s\int d^3\vec{p}\,\frac{m}{\omega}\Big[\hat{a}_s(\vec{p})u_s(\vec{p})\,e^{-i(\omega-\omega')t}\,\delta^3(\vec{p}-\vec{p}') \\
&\qquad + \hat{b}_s^\dagger(\vec{p})v_s(\vec{p})e^{i\,(\omega'+\omega)t}\,\delta^3(\vec{p}'+\vec{p})\Big]
\end{aligned}
$$

Acum, putem face şi integrala după p

$$\boxed{\int d^3\vec{x}\,\hat{\varphi}(\vec{x},t)\,e^{i\,p'\cdot x}} = \sum_s \frac{m}{\omega}\left[\hat{a}_s(\vec{p}\,')\,u_s(\vec{p}\,') + \hat{b}_s^\dagger(-\vec{p}\,')\,v_s(-\vec{p}\,')\,e^{i\,2\omega' t}\right]$$

$$p' \to p \qquad \boxed{\int d^3\vec{x}\,\hat{\varphi}(\vec{x},t)\,e^{i\,p\cdot x}} = \sum_s \frac{m}{\omega}\left[\hat{a}_s(\vec{p})\,u_s(\vec{p}) + \hat{b}_s^\dagger(-\vec{p})\,v_s(-\vec{p})\,e^{i\,2\omega t}\right]$$

$$p \to -p \qquad \boxed{\int d^3\vec{x}\,\hat{\varphi}(\vec{x},t)\,e^{-i\,p\cdot x}} = \sum_s \frac{m}{\omega}\left[\hat{a}_s(-\vec{p})\,u_s(-\vec{p}) + \hat{b}_s^\dagger(\vec{p})\,v_s(\vec{p})\,e^{i\,2\omega t}\right]$$

Proiectăm mai departe pe $\overline{u}_r$

$$\boxed{\int d^3\vec{x}\,\overline{u}_r(\vec{p})\,\hat{\varphi}(\vec{x},t)\,e^{-i\,p\cdot x}} = \sum_s \frac{m}{\omega}\left[\hat{a}_s(-\vec{p})\,\overline{u}_r(\vec{p})\,u_s(-\vec{p}) + \hat{b}_s^\dagger(\vec{p})\,\underbrace{\overline{u}_r(\vec{p})\,v_s(\vec{p})}_{=0\,(1.8)}\,e^{i\,2\omega t}\right]$$

$$\boxed{\int d^3\vec{x}\;\underbrace{\overline{u}_r(\vec{p})}\,\hat{\varphi}(\vec{x},t)\,e^{-i\,p\cdot x} = \sum_s \frac{m}{\omega}\hat{a}_s(-\vec{p})\,\overline{u}_r(\vec{p})\;\underbrace{u_s(-\vec{p})}}$$

$$\overline{u}_r(\vec{p}) = \overline{u}_r(-\vec{p})\gamma^0 \quad ; \quad u_s(-\vec{p}) = \gamma^0 u_s(\vec{p})$$

$$\boxed{\int d^3\vec{x}\;\overbrace{\overline{u}_r(-\vec{p})\,\gamma^0}\,\hat{\varphi}(\vec{x},t)\,e^{-i\,p\cdot x}} = \sum_s \frac{m}{\omega}\hat{a}_s(-\vec{p})\;\underbrace{\overline{u}_r(\vec{p})\;\overbrace{\gamma^0 u_s(\vec{p})}}_{=\frac{\omega}{m}\,\delta_{rs}}$$

din (1.9) avem $\overline{u}_r(\vec{p})\,\gamma^0 u_s(\vec{p}) = \dfrac{\omega}{m}\delta_{rs}$

$$\boxed{\int d^3\vec{x}\;\overline{u}_r(-\vec{p})\,\gamma^0\hat{\varphi}(\vec{x},t)\,e^{-i\,p\cdot x}} = \sum_s \hat{a}_s(-\vec{p})\,\delta_{rs} = \hat{a}_r(-\vec{p})$$

Operatorul căutat, ca amplitudinea $\hat{a}_r(\vec{p})$ a dezvoltării Fourier (1.11), este:

$$\boxed{\hat{a}_r(\vec{p}) = \int d^3\vec{x}\,\overline{u}_r(\vec{p})\,\gamma^0\hat{\varphi}(\vec{x},t)\,e^{i\,p\cdot x}} \tag{1.13}$$

sau $\hat{a}_r(\vec{p}) = \displaystyle\int d^3\vec{x}\,u_r^\dagger(\vec{p})\,\hat{\varphi}(\vec{x},t)\,e^{i\,p\cdot x}$

Operatorul hermitic conjugat $\hat{a}_r^\dagger(\vec{p})$ este:

$$\hat{a}_r^\dagger(\vec{p}) = \int d^3\vec{x}\,\hat{\varphi}^\dagger(\vec{x},t)\,u_r(\vec{p})\,e^{-i\,p\cdot x} \tag{1.14}$$

$$\boxed{\hat{a}_r^\dagger(\vec{p}) = \int d^3\vec{x}\;\hat{\overline{\varphi}}(\vec{x},t)\,\gamma^0 u_r(\vec{p})\,e^{-i\,p\cdot x}} \tag{1.15}$$

Determinarea amplitudinilor $\hat{b}_s(\vec{p})$

- Dezvoltarea Fourier (1.7) a câmpului adjunct $\hat{\overline{\varphi}}(\vec{x},t)$, este:

$$\hat{\overline{\varphi}}(\vec{x},t) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\left[\hat{b}_s(\vec{p})\overline{v}_s(\vec{p})e^{-i\,p\cdot x} + \hat{a}_s^\dagger(\vec{p})\overline{u}_s(\vec{p})e^{i\,\vec{p}\cdot x}\right] \tag{1.16}$$

Proiectăm (1.16) pe componenta armonică $e^{i\,p'\cdot x}$ de funcţii ortogonale:

$$\boxed{\int d^3\vec{x}\,\hat{\overline{\varphi}}(\vec{x},t)e^{i\,p'\cdot x}} = \sum_s \int d^3\vec{x}\frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\left[\hat{b}_s(\vec{p})\overline{v}_s(\vec{p})e^{-i(p-p')\cdot x} + \hat{a}_s^\dagger(\vec{p})\overline{u}_s(\vec{p})e^{i\,(p+p')\cdot x}\right]$$

$$= \sum_s \int d^3\vec{x}\frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\left[\hat{b}_s(\vec{p})\overline{v}_s(\vec{p})\,e^{-i\,(\omega-\omega')t}\,e^{i\,(\vec{p}-\vec{p}')\cdot\vec{x}} + \right.$$

$$\left. + \hat{a}_s^\dagger(\vec{p})\overline{u}_s(\vec{p})\,e^{i\,(\omega'+\omega)t}\,e^{-i\,(\vec{k}'+\vec{k})\cdot\vec{x}}\right]$$

Pentru integrarea după x, folosim expresiile (1.12), cu funcţia δ

$$\int \frac{d^3\vec{x}}{(2\pi)^3}e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = \delta^3(\vec{k}-\vec{k}') \quad ; \quad \int \frac{d^3\vec{x}}{(2\pi)^3}e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} = \delta^3(\vec{k}'+\vec{k}) \tag{1.12}$$

Atunci, integrala de mai sus după $d^3\vec{x}$ va fi:

$$\boxed{\int d^3\vec{x}\,\hat{\overline{\varphi}}(\vec{x},t)e^{i\,p'\cdot x}} = \sum_s \int d^3\vec{p}\,\frac{m}{\omega}\left[\hat{b}_s(\vec{p})\overline{v}_s(\vec{p})\,e^{-i\,(\omega-\omega')t}\,\delta^3(\vec{p}-\vec{p}') + \right.$$

$$\left. + \hat{a}_s^\dagger(\vec{p})\overline{u}_s(\vec{p})\,e^{i\,(\omega'+\omega)t}\,\delta^3(\vec{p}'+\vec{p})\right]$$

Acum, facem şi integrala după p

$$\boxed{\int d^3\vec{x}\,\hat{\overline{\varphi}}(\vec{x},t)\,e^{i\,p'\cdot x}} = \sum_s \frac{m}{\omega}\left[\hat{b}_s(\vec{p}')\,\overline{v}_s(\vec{p}') + \hat{a}_s^\dagger(-\vec{p}')\,\overline{u}_s(-\vec{p}')\,e^{i\,2\omega' t}\right]$$

$$\boxed{\int d^3\vec{x}\,\hat{\overline{\varphi}}(\vec{x},t)\,e^{i\,p\cdot x}} = \sum_s \frac{m}{\omega}\left[\hat{b}_s(\vec{p})\,\overline{v}_s(\vec{p}) + \hat{a}_s^\dagger(-\vec{p})\,\overline{u}_s(-\vec{p})\,e^{i\,2\omega t}\right]$$

$$\boxed{\int d^3\vec{x}\,\hat{\overline{\varphi}}(\vec{x},t)\,e^{-i\,p\cdot x}} = \sum_s \frac{m}{\omega}\left[\hat{b}_s(-\vec{p})\,\overline{v}_s(-\vec{p}) + \hat{a}_s^\dagger(\vec{p})\,\overline{u}_s(\vec{p})\,e^{i\,2\omega t}\right]$$

Proiectăm mai departe pe $v_r(\vec{p})$

$$\boxed{\int d^3\vec{x}\;\hat{\overline{\varphi}}(\vec{x},t)\,v_r(\vec{p})\,e^{-i\,p\cdot x}} = \sum_s \frac{m}{\omega}\left[\hat{b}_s(-\vec{p})\,\overline{v}_s(-\vec{p})v_r(\vec{p}) + \hat{a}_s^\dagger(\vec{p})\,\underbrace{\overline{u}_s(\vec{p})\,v_r(\vec{p})}_{=0\,(1.8)}\,e^{i\,2\omega t}\right]$$

$$\boxed{\int d^3\vec{x}\ \hat{\overline{\varphi}}(\vec{x},t)\,\underbrace{v_r(\vec{p})}\,e^{-i\,p\cdot x}} = \sum_s \frac{m}{\omega}\hat{b}_s(-\vec{p})\ \underbrace{\overline{v}_s(-\vec{p})}\,v_r(\vec{p})$$

$$v_r(\vec{p}) = -\gamma^0 v_r(-\vec{p}) \quad ; \quad \overline{v}_s(-\vec{p}) = -\overline{v}_s(\vec{p})\gamma^0$$

$$\boxed{-\int d^3\vec{x}\ \hat{\overline{\varphi}}(\vec{x},t)\,\overbrace{\gamma^0 v_r(-\vec{p})}\,e^{-i\,p\cdot x}} = -\sum_s \frac{m}{\omega}\hat{b}_s(-\vec{p})\ \underbrace{\overbrace{\overline{v}_s(\vec{p})\,\gamma^0}\,v_r(\vec{p})}_{=\frac{\omega}{m}\,\delta_{sr}}$$

din (1.10) avem $\overline{v}_r(\vec{p})\,\gamma^0 v_s(\vec{p}) = \dfrac{\omega}{m}\delta_{rs}$

$$\boxed{\int d^3\vec{x}\ \hat{\overline{\varphi}}(\vec{x},t)\gamma^0\,v_r(-\vec{p})\,e^{-i\,p\cdot x}} = \sum_s \hat{b}_s(-\vec{p})\,\delta_{rs} = \hat{b}_r(-\vec{p})$$

$$\int d^3\vec{x}\ \hat{\overline{\varphi}}(\vec{x},t)\,\gamma^0\,v_r(\vec{p})\,e^{i\,p\cdot x} = \hat{b}_r(\vec{p})$$

$$\boxed{\hat{b}_r(\vec{p}) = \int d^3\vec{x}\,\overline{\varphi}(\vec{x},t)\gamma^0 v_r(\vec{p})\,e^{i\,p\cdot x}} \tag{1.17}$$

sau $\quad \hat{b}_r(\vec{p}) = \displaystyle\int d^3\vec{x}\,\hat{\varphi}^\dagger(\vec{x},t)\,v_r(\vec{p})\,e^{i\,p\cdot x}$

Operatorul hermitic conjugat $\hat{b}_r^\dagger(\vec{p})$ este:

$$\hat{b}_r^\dagger(\vec{p}) = \int d^3\vec{x}\,v_r^\dagger(\vec{p})\ \hat{\varphi}(\vec{x},t)\,e^{-i\,p\cdot x}$$

$$\boxed{\hat{b}_r^\dagger(\vec{p}) = \int d^3x\,\overline{v}_r(\vec{p})\,\gamma^0\hat{\varphi}(\vec{x},t)\,e^{-i\,p\cdot x}} \tag{1.18}$$

- In concluzie, setul complet de operatori căutaţi, este:

$$\boxed{\begin{aligned}
\hat{a}_r(\vec{p}) &= \int d^3x\,\overline{u}_r(\vec{p})\,\gamma^0\hat{\varphi}(\vec{x},t)\,e^{i\,p\cdot x}\\
\hat{a}_r^\dagger(\vec{p}) &= \int d^3x\,\hat{\overline{\varphi}}(\vec{x},t)\,\gamma^0 u_r(\vec{p})\,e^{-i\,p\cdot x}\\
\hat{b}_r(\vec{p}) &= \int d^3x\,\overline{\varphi}(\vec{x},t)\gamma^0 v_r(\vec{p})\,e^{i\,p\cdot x}\\
\hat{b}_r^\dagger(\vec{p}) &= \int d^3x\,\overline{v}_r\vec{p})\,\gamma^0\hat{\varphi}(\vec{x},t)\,e^{-i\,p\cdot x}
\end{aligned}} \tag{1.19}$$

1.1.2 Anticomutare operatori de câmp Dirac

Cuantificarea câmpurilor Dirac folosind *relaţiile de comutare* ale operatorilor la acelaşi timp, conduce la rezultate nefizice. In schimb, dacă se folosesc *relaţiile de anticomutare* ale aceloraşi operatori, rezultatele sunt cele fizice. Aceasta ne arată că ori de câte ori avem de cuantificat *câmpuri cu spin semiîntreg* trebuie să apelăm la relaţiile de anticomutare. Aceasta face ca aceste câmpuri să aibe cuante care se supun statisticii Fermi-Dirac, adică să fie fermioni, după cum vom vedea în continuare.

Invers, încercarea de a cuantifica *câmpurile de spin întreg* folosind relaţiile de anticomutare a operatorilor la acelaşi timp, conduce la rezultate nefizice.
Această legătură exprimă teorema spin - statistică.

Inlocuind relaţiile de comutare de la cuantificarea câmpului scalar sau chiar a oscilatorului armonic, cu *relaţii de anticomutare* pentru opertorii de câmp Dirac, avem:

$$\begin{aligned}
&\left\{\hat{\varphi}_i(\vec{x},t),\hat{\pi}_{\varphi_j}(\vec{x}\,')\right\} = i\,\delta^3(\vec{x}-\vec{x}\,')\,\delta_{ij}\\
&\left\{\hat{\varphi}_i(\vec{x},t),\hat{\varphi}^\dagger_j(\vec{x}\,',t)\right\} = \delta^3(\vec{x}-\vec{x}\,')\,\delta_{ij}\\
&\left\{\hat{\pi}_{\varphi_i}(\vec{x},t),\hat{\pi}_{\varphi_j}(\vec{x}\,',t)\right\} = 0\\
&\left\{\hat{\varphi}_i(\vec{x},t),\hat{\varphi}_j(\vec{x}\,',t)\right\} = 0
\end{aligned} \tag{1.20}$$

Relaţii de anticomutare $\left\{\hat{a}_r(\vec{p}),\hat{a}^\dagger_s(\vec{p}\,')\right\}$

- Să evaluăm relaţiile de anticomutare $\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}^\dagger_s(\vec{p}\,')\right\}}$ folosind expresiile amplitudinilor Fourier (1.19) $\hat{a}_r(\vec{p})$ şi $\hat{a}^\dagger_s(\vec{p})$ prin operatorii de câmp Dirac.

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}^\dagger_s(\vec{p}\,')\right\}} = \int d^3\vec{x}\;d^3\vec{x}\,'\;e^{i\,(p\cdot x-p'\cdot x')}$$
$$\Big[\overbrace{\overline{u}_r(\vec{p})\,\gamma^0\hat{\varphi}(\vec{x},t)}^{\hat{a}_r}\;\overbrace{\hat{\overline{\varphi}}(\vec{x}\,',t)\,\gamma^0 u_s(\vec{p}\,')}^{\hat{a}^\dagger_s}+\overbrace{\hat{\overline{\varphi}}(\vec{x}\,',t)\,\gamma^0 u_s(\vec{p}\,')}^{\hat{a}^\dagger_s}\,\overbrace{\overline{u}_r(\vec{p})\,\gamma^0\hat{\varphi}(\vec{x},t)}^{\hat{a}_r}\Big]$$

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}^\dagger_s(\vec{p}\,')\right\}} = \int d^3\vec{x}\;d^3\vec{x}\,'\;e^{i\,(p\cdot x-p'\cdot x')}$$
$$\Big[\overbrace{u^\dagger_r(\vec{p})\,\hat{\varphi}(\vec{x},t)}^{\hat{a}_r}\;\overbrace{\hat{\varphi}^\dagger(\vec{x}\,',t)\,u_s(\vec{p}\,')}^{\hat{a}^\dagger_s}+\overbrace{\hat{\varphi}^\dagger(\vec{x}\,',t)\,u_s(\vec{p}\,')}^{\hat{a}^\dagger_s}\,\overbrace{u^\dagger_r(\vec{p})\,\hat{\varphi}(\vec{x},t)}^{\hat{a}_r}\Big]$$

unde am folosit $t=t'$.

Rescriind relaţiile de anticomutare de mai sus, explicit cu indicii de sumare i,j după componentele de câmp, avem:

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\ d^3\vec{x}\,'\ e^{i\,(p\cdot x - p'\cdot x')}$$

$$\Big[\overbrace{\underbrace{u_{ri}^\dagger(\vec{p})}\ \hat{\varphi}_i(\vec{x},t)}^{\hat{a}_r}\ \overbrace{\hat{\varphi}_j^\dagger(\vec{x}\,',t)\ \underbrace{u_{sj}(\vec{p}\,')}}^{\hat{a}_s^\dagger} + \overbrace{\hat{\varphi}_j^\dagger(\vec{x}\,',t)\ \underbrace{u_{sj}(\vec{p}\,')}}^{\hat{a}_s^\dagger}\overbrace{\underbrace{u_{ri}^\dagger(\vec{p})}\ \hat{\varphi}_i(\vec{x},t)}^{\hat{a}_r}\Big]$$

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\ d^3\vec{x}\,'\ e^{i\,(p\cdot x - p'\cdot x')}$$

$$\overbrace{u_{ri}^\dagger(\vec{p})}\ \Big[\underbrace{\hat{\varphi}_i(\vec{x},t)\,\hat{\varphi}_j^\dagger(\vec{x}\,',t) + \hat{\varphi}_j^\dagger(\vec{x}\,',t)\,\hat{\varphi}_i(\vec{x},t)}\Big]\ \overbrace{u_{sj}(\vec{p}\,')}$$

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\ d^3\vec{x}\,'\ e^{i\,(p\cdot x - p'\cdot x')}\ \overbrace{u_{ri}^\dagger(\vec{p})}\ \underbrace{\left\{\hat{\varphi}_i(\vec{x},t),\hat{\varphi}_j^\dagger(\vec{x}\,',t)\right\}}_{\delta^3(\vec{x}-\vec{x}\,')\delta_{ij}\,(1.20)}\ \overbrace{u_{sj}(\vec{p}\,')}$$

Folosind relaţiile de <u>anticomutare a operatorilor la acelaşi timp</u> (1.20), obţinem:

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\ d^3\vec{x}\,'\ e^{i\,(p\cdot x - p'\cdot x')}\ u_{ri}^\dagger(\vec{p})\ \delta^3(\vec{x}-\vec{x}\,')\ \underbrace{\delta_{ij}\,u_{sj}(\vec{p}\,')}_{u_{si}(\vec{p})}$$

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\ d^3\vec{x}\,'\ e^{i\,(p\cdot x - p'\cdot x')}\ u_{ri}^\dagger(\vec{p})\ \overbrace{u_{si}(\vec{p}\,')}\ \delta^3(\vec{x}-\vec{x}\,')$$

suprimând indicii de câmp de la spinor:

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\ d^3\vec{x}\,'\ e^{i\,(p\cdot x - p'\cdot x')}\ u_r^\dagger(\vec{p})\,u_s(\vec{p}\,')\,\delta^3(\vec{x}-\vec{x}\,')$$

Efectuând integrarea după $d^3\vec{x}\,'$, obţinem:

$$\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\} = \underbrace{\int d^3\vec{x}\ e^{i\,(p\,-\,p')\cdot x}}\ u_r^\dagger(\vec{p})\,u_s(\vec{p}\,')$$

Deoarece $E=\hbar\omega$ şi $\vec{p}=\hbar\vec{k}$ cu $E^2=\vec{p}^{\,2}c^2+m^2c^4$, pentru integrarea după $d^3\vec{x}$, folosim

$$\int d^3\vec{x}\ e^{i\,(k\,-\,k')\cdot x} = \int \underbrace{d^3\vec{x}}\ \ e^{i\,(\omega\,-\,\omega')\,t}\ \underbrace{e^{-i\,(\vec{k}\,-\,\vec{k}\,')\cdot\vec{x}}}$$

$$= e^{i\,(\omega\,-\,\omega')\,t}\ \overbrace{(2\pi)^3\,\delta^3(\vec{k}-\vec{k}\,')} = (2\pi)^3\delta^3(\vec{k}-\vec{k}\,')$$

In acest fel, relaţia de anticomutare este:

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\}} = (2\pi)^3\delta^3(\vec{p}-\vec{p}\,')\ \underbrace{u_r^\dagger(\vec{p})\,u_s(\vec{p})}$$

Folosind identitatea (1.9): $\quad u_r^\dagger(p)\,u_s(p) = \dfrac{\omega}{m}\delta_{rs},\quad$ avem,

$$\boxed{\left\{\hat{a}_r(\vec{p}),\hat{a}_s^\dagger(\vec{p}\,')\right\} = (2\pi)^3\,\frac{\omega}{m}\,\delta^3(\vec{p}-\vec{p}\,')\,\delta_{rs}} \tag{1.21}$$

Relaţii de anticomutare $\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}$

- Să evaluăm relaţiile de anticomutare $\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}}$, folosind expresiile amplitudinilor Fourier (1.19) $\hat{b}_r(\vec{p})$ şi $\hat{b}_s^\dagger(\vec{p})$ prin operatorii de câmp Dirac.

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(p\cdot x - p'\cdot x\,')}$$
$$\left[\overbrace{\hat{\overline{\varphi}}(\vec{x},t)\,\gamma^0\,v_r(\vec{p})}^{\hat{b}_r}\;\overbrace{\overline{v}_s(\vec{p}\,')\,\gamma^0\hat{\varphi}(\vec{x}\,',t)}^{\hat{b}_s^\dagger} + \overbrace{\overline{v}_s(\vec{p}\,')\,\gamma^0\hat{\varphi}(\vec{x}\,',t)}^{\hat{b}_s^\dagger}\;\overbrace{\hat{\overline{\varphi}}(\vec{x},t)\,\gamma^0 v_r(\vec{p})}^{\hat{b}_r}\right]$$

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(p\cdot x - p'\cdot x')}$$
$$\left[\overbrace{\hat{\varphi}^\dagger(\vec{x},t)\,v_r(\vec{p})}\;\overbrace{v_s^\dagger(\vec{p}\,')\,\hat{\varphi}(\vec{x}\,',t)} + \overbrace{v_s^\dagger(\vec{p}\,')\,\hat{\varphi}(\vec{x}\,',t)}\;\overbrace{\hat{\varphi}^\dagger(\vec{x},t)\,v_r(\vec{p})}\right]$$

unde am folosit $t = t'$.

Rescriind relaţiile de anticomutare de mai sus, explicit cu indicii de sumare i, j după componentele de câmp, avem:

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(p\cdot x - p'\cdot x')}$$
$$\left[\overbrace{\hat{\varphi}_j^\dagger(\vec{x},t)\,\underbrace{v_{rj}(\vec{p})}}\;\overbrace{\underbrace{v_{si}^\dagger(\vec{p}\,')}\,\hat{\varphi}_i(\vec{x}\,',t)} + \overbrace{\underbrace{v_{si}^\dagger(\vec{p}\,')}\,\hat{\varphi}_i(\vec{x}\,',t)}\;\overbrace{\hat{\varphi}_j^\dagger(\vec{x},t)\,\underbrace{v_{rj}(\vec{p})}}\right]$$

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(p\cdot x - p'\cdot x')}$$
$$\overbrace{v_{si}^\dagger(\vec{p}\,')}\left[\underbrace{\hat{\varphi}_j^\dagger(\vec{x},t)\,\hat{\varphi}_i(\vec{x}\,',t) + \hat{\varphi}_i(\vec{x}\,',t)\,\hat{\varphi}_j^\dagger(\vec{x},t)}\right]\overbrace{v_{rj}(\vec{p})}$$

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(p\cdot x - p'\cdot x')}\;\overbrace{v_{si}^\dagger(\vec{p}\,')}\;\underbrace{\left\{\overbrace{\hat{\varphi}_i(\vec{x}\,',t), \hat{\varphi}_j^\dagger(\vec{x},t)}\right\}}_{\delta^3(\vec{x}\,'-\vec{x})\delta_{ji}\ (1.20)}\;\overbrace{v_{rj}(\vec{p})}$$

Folosind relaţiile de anticomutare a operatorilor la acelaşi timp (1.20), obţinem:

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(p\cdot x - p'\cdot x')}\;\overbrace{v_{si}^\dagger(\vec{p}\,')}\;\delta^3(\vec{x}\,' - \vec{x})\;\underbrace{\delta_{ji}\,\overbrace{v_{rj}(\vec{p})}}_{v_{ri}(\vec{p})}$$

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(p\cdot x - p'\cdot x')}\;v_{si}^\dagger(\vec{p}\,')\;\overbrace{v_{ri}(\vec{p})}\;\delta^3(\vec{x}\,' - \vec{x})$$

suprimând indicele i de câmp de la spinori,

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,'\,')\right\}} = \int d^3\vec{x}\;\underbrace{d^3\vec{x}\,'\; e^{i\,(p\cdot x - p'\cdot x')}}\;v_s^\dagger(\vec{p}\,')\,v_r(\vec{p})\;\underbrace{\delta^3(\vec{x}\,' - \vec{x})}$$

Efectuând integrarea după $d^3\vec{x}\,'$, obţinem:

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = \int d^3\vec{x}\; e^{i\,(p\,-\,p')\cdot x}\; v_s^\dagger(\vec{p}\,')\, v_r(\vec{p})$$

Pentru integrarea după $d^3\vec{x}$ folosim,

$$\int d^3\vec{x}\; e^{i\,(k\,-\,k')\cdot x} = \int \underbrace{d^3\vec{x}}\; e^{i\,(\omega\,-\,\omega')\,t}\; \underbrace{e^{-i\,(\vec{k}\,-\,\vec{k}\,')\cdot\vec{x}}}$$

$$= e^{i\,(\omega\,-\,\omega')\,t}\; \overbrace{(2\pi)^3\,\delta^3(\vec{k} - \vec{k}\,')} = (2\pi)^3\delta^3(\vec{k} - \vec{k}\,')$$

In acest fel, relaţia de anticomutare este:

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\}} = (2\pi)^3\delta^3(\vec{p} - \vec{p}\,')\; \underbrace{v_s^\dagger(\vec{p})\, v_r(\vec{p})}$$

Folosind identitatea (1.10): $v_r^\dagger(\vec{p})\, v_s(\vec{p}) = \dfrac{\omega}{m}\delta_{rs}$, avem,

$$\boxed{\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\} = (2\pi)^3\,\frac{\omega}{m}\,\delta^3(\vec{p} - \vec{p}\,')\,\delta_{rs}} \tag{1.22}$$

- Ceilalţi anticomutatori între coeficienţii Fourier a şi b sunt egali cu zero, bazat pe relaţiile de anticomutare pentru operatorii de câmp (1.20):

$$\left\{\hat{\pi}_{\hat{\varphi}_i}(\vec{x}, t), \hat{\pi}_{\varphi_j}(\vec{x}\,', t)\right\} = 0 \quad ; \quad \{\hat{\varphi}_i(\vec{x}, t), \hat{\varphi}_j(\vec{x}\,', t)\} = 0$$

- In rezumat, setul complet de relaţii de anticomutare pentru coeficienţii Fourier este:

$$\boxed{\begin{array}{l}
\left\{\hat{a}_r(\vec{p}), \hat{a}_s^\dagger(\vec{p}\,')\right\} = (2\pi)^3\,\dfrac{\omega}{m}\,\delta^3(\vec{p} - \vec{p}\,')\,\delta_{rs} \\
\left\{\hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\} = (2\pi)^3\,\dfrac{\omega}{m}\,\delta^3(\vec{p} - \vec{p}\,')\,\delta_{rs} \\
\left\{\hat{a}_r(\vec{p}), \hat{a}_s(\vec{p}\,')\right\} = 0 \quad ; \quad \left\{\hat{a}_r^\dagger(\vec{p}), \hat{a}_s^\dagger(\vec{p}\,')\right\} = 0 \\
\left\{\hat{a}_r(\vec{p}), \hat{b}_s(\vec{p}\,')\right\} = 0 \quad ; \quad \left\{\hat{a}_r^\dagger(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\} = 0 \\
\left\{\hat{a}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,')\right\} = 0 \quad ; \quad \left\{\hat{a}_r^\dagger(\vec{p}), \hat{b}_s(\vec{p}\,')\right\} = 0
\end{array}} \tag{1.23}$$

1.1.3 Hamiltonian de câmp Dirac cuantic

- Având stabilite relaţiile de anticomutare, le putem utiliza în dezvoltările cuantice de câmpuri Dirac. La început vom exprima Hamiltonian-ul de câmp cuantic Dirac în termeni de coeficienţi Fourier (operatori de creare şi anihilare).

$$H_D = \int d^3\vec{x}\, i\hbar\, \hat{\varphi}^\dagger\, \partial_0 \hat{\varphi} \tag{1.24}$$

unde

$$\hat{\varphi}(\vec{x},t) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{m}{\omega} \Big(\hat{a}_s(\vec{p})\, u_s(\vec{p})\, e^{-i\,p\cdot x} + \hat{b}_s^\dagger(\vec{p})\, v_s(\vec{p})\, e^{i\,p\cdot x} \Big)$$

$$\begin{aligned}
\partial_0 \hat{\varphi}(\vec{x},t) &= \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{m}{\omega} (-i\omega) \Big[\hat{a}_s(\vec{p})\, u_s(\vec{p})\, e^{-i\,p\cdot x} - \hat{b}_s^\dagger(\vec{p})\, v_s(\vec{p})\, e^{i\,p\cdot x} \Big] \\
\hat{\varphi}^\dagger(\vec{x},t) &= \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{m}{\omega} \Big[\hat{b}_s(\vec{p})\, v_s^\dagger(\vec{p})\, e^{-i\,p\cdot x} + \hat{a}_s^\dagger(\vec{p})\, u_s^\dagger(\vec{p})\, e^{i\,p\cdot x} \Big]
\end{aligned}$$

Inlocuind în (1.24) obţinem:

$$\begin{aligned}
H_D = \sum_{r,s} \int & d^3\vec{x}\, \overbrace{\frac{d^3\vec{p}\,'}{(2\pi)^3}\, \frac{m}{\omega'} \Big[\hat{b}_r(\vec{p})\, v_r^\dagger(\vec{p})\, e^{-i\,p\cdot x} + \hat{a}_r^\dagger(\vec{p})\, u_r^\dagger(\vec{p})\, e^{i\,p\cdot x} \Big]}^{\hat{\varphi}^\dagger(\vec{x},t)} \times \\
& \times \overbrace{\frac{d^3\vec{p}}{(2\pi)^3}\, \frac{m}{\omega} (-i\omega') \Big[\hat{a}_s(\vec{p}\,')\, u_s(\vec{p}\,')\, e^{-i\,p'\cdot x} - \hat{b}_s^\dagger(\vec{p}\,')\, v_s(\vec{p}\,')\, e^{i\,p'\cdot x} \Big]}^{\partial_0 \hat{\varphi}}
\end{aligned}$$

$$\begin{aligned}
H_D = \sum_{r,s} \int & d^3\vec{x} \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega} \frac{d^3\vec{p}\,'}{(2\pi)^3} \frac{m}{\omega'}\, \omega' \times \\
\times \Big[& \hat{b}_r(\vec{p})\, \hat{a}_s(\vec{p}\,')\, v_r^\dagger(\vec{p})\, u_s(\vec{p}\,')\, e^{-i\,(p+p')\cdot x} - \hat{b}_r(\vec{p})\, \hat{b}_s^\dagger(\vec{p}\,')\, v_r^\dagger(\vec{p})\, v_s(\vec{p}\,')\, e^{-i\,(p-p')\cdot x} \\
& + \hat{a}_r^\dagger(\vec{p})\, \hat{a}_s(\vec{p}\,')\, u_r^\dagger(\vec{p})\, u_s(\vec{p}\,')\, e^{i\,(p-p')\cdot x} - \hat{a}_r^\dagger(\vec{p})\, \hat{b}_s^\dagger(\vec{p}\,')\, u_r^\dagger(\vec{p})\, v_s(\vec{p}\,')\, e^{i\,(p+p')\cdot x} \Big]
\end{aligned}$$

Pentru a face integrarea după $d^3\vec{x}$ folosim relaţiile următoare:

$$\begin{aligned}
\int d^3\vec{x}\, e^{-i\,(k-k')\cdot x} &= \int d^3\vec{x}\, e^{-i\,(\omega-\omega')t}\, e^{i\,(\vec{k}-\vec{k}\,')\cdot\vec{x}} = e^{-i\,(\omega-\omega')t}\, (2\pi)^3\, \delta^3(\vec{k}-\vec{k}\,') \\
&= (2\pi)^3\, \delta^3(\vec{k}-\vec{k}\,') \\
\int d^3\vec{x}\, e^{-i\,(k+k')\cdot x} &= \int d^3\vec{x}\, e^{-i\,(\omega+\omega')t}\, e^{i\,(\vec{k}+\vec{k}\,')\cdot\vec{x}} = e^{-i\,(\omega+\omega')t}\, (2\pi)^3\, \delta^3(\vec{k}+\vec{k}\,') \\
&= e^{-2\,i\,\omega\,t} (2\pi)^3\, \delta^3(\vec{k}+\vec{k}\,')
\end{aligned}$$

Inlocuind în expresiile dinainte ale Hamiltonian-ului, după integrarea $d^3\vec{x}$, obţinem:

$$
\begin{aligned}
H_D = \sum_{r,s} \int & \frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega' \times \\
& \times \Big[\hat{b}_r(\vec{p})\,\hat{a}_s(\vec{p}\,')\,v_r^\dagger(\vec{p})\,u_s(\vec{p}\,')\,\overbrace{e^{-2i\,\omega t}(2\pi)^3\delta^3(\vec{p}+\vec{p}\,')} \\
& \quad -\hat{b}_r(\vec{p})\,\hat{b}_s^\dagger(\vec{p}\,')\,v_r^\dagger(\vec{p})\,v_s(\vec{p}\,')\,\overbrace{(2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')} \\
& \quad +\hat{a}_r^\dagger(\vec{p})\,\hat{a}_s(\vec{p}\,')\,u_r^\dagger(\vec{p})\,u_s(\vec{p}\,')\,\overbrace{(2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')} \\
& \quad -\hat{a}_r^\dagger(\vec{p})\,\hat{b}_s^\dagger(\vec{p}\,')\,u_r^\dagger(\vec{p})\,v_s(\vec{p}\,')\,\overbrace{e^{2i\,\omega t}(2\pi)^3\,\delta^3(\vec{p}+\vec{p}\,')}\Big]
\end{aligned}
$$

Acum prin integrarea $d^3\vec{p}\,'$, obţinem

$$
\begin{aligned}
H_D = \sum_{r,s} \int & \frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\frac{m}{\omega}\,\omega \times \\
& \times \Big[\hat{b}_r(\vec{p})\,\hat{a}_s(-\vec{p})\,v_r^\dagger(\vec{p})\,\underbrace{u_s(-\vec{p})}_{=\gamma^0 u_s(\vec{p})}\,e^{-2i\,\omega t} - \hat{b}_r(\vec{p})\,\hat{b}_s^\dagger(\vec{p})\,v_r^\dagger(\vec{p})\,v_s(\vec{p}) \\
& \quad + \hat{a}_r^\dagger(\vec{p})\,\hat{a}_s(\vec{p})\,u_r^\dagger(\vec{p})\,u_s(\vec{p}) - \hat{a}_r^\dagger(\vec{p})\,\hat{b}_s^\dagger(-\vec{p})\,u_r^\dagger(\vec{p})\,\underbrace{v_s(-\vec{p})}_{=\gamma^0 v_s(\vec{p})}\,e^{2i\,\omega t}\Big]
\end{aligned}
$$

deoarece, $\quad -\gamma^0 v_r(\vec{p}) = v_r(-\vec{p}) \quad$ şi $\quad u_s(-\vec{p}) = \gamma^0 u_s(\vec{p})$

$$
\begin{aligned}
H_D = \sum_{r,s} \int & \frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\frac{m}{\omega}\,\omega \times \\
& \times \Big[\hat{b}_r(\vec{p})\,\hat{a}_s(-\vec{p})\,\underbrace{\overbrace{\overline{v}_r(\vec{p})}^{v_r^\dagger(\vec{p})\gamma^0}\,u_s(\vec{p})}_{=0}\,e^{-2i\,\omega t} - \hat{b}_r(\vec{p})\,\hat{b}_s^\dagger(\vec{p})\,v_r^\dagger(\vec{p})\,v_s(\vec{p}) \\
& \quad + \hat{a}_r^\dagger(\vec{p})\,\hat{a}_s(\vec{p})\,u_r^\dagger(\vec{p})\,u_s(\vec{p}) + \hat{a}_r^\dagger(\vec{p})\,\hat{b}_s^\dagger(-\vec{p})\,\underbrace{\overline{u}_r(\vec{p})\,v_s(\vec{p})}_{=0}\,e^{2i\,\omega\,t}\Big]
\end{aligned}
$$

deoarece, $\quad \overline{v}_r(\vec{p})u_s(\vec{p}) = 0 \quad$ şi $\quad \overline{u}_r(\vec{p})v_s(\vec{p}) = 0$

$$
H_D = \sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{m}{\omega}\frac{m}{\omega}\,\omega\Big[\hat{a}_r^\dagger(\vec{p})\,\hat{a}_s(\vec{p})\,\underbrace{u_r^\dagger(\vec{p})\,u_s(\vec{p})}_{=\frac{\omega}{m}\delta_{rs}} - \hat{b}_r(\vec{p})\,\hat{b}_s^\dagger(\vec{p})\,\underbrace{v_r^\dagger(\vec{p})\,v_s(\vec{p})}_{=\frac{\omega}{m}\delta_{rs}}\Big]
$$

deoarece, $\quad u_r^\dagger(\vec{p})\,u_s(\vec{p}) = \dfrac{\omega}{m}\delta_{rs} \quad (1.9)$

şi $\quad v_r^\dagger(\vec{p})\,v_s(\vec{p}) = \dfrac{\omega}{m}\delta_{rs} \quad (1.10)$

$$\begin{aligned} H_D &= \sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega} \frac{m}{\omega}\, \omega \left[\hat{a}_r^\dagger(\vec{p})\, \hat{a}_s(\vec{p})\, \frac{\omega}{m} \delta_{rs} - \hat{b}_r(\vec{p})\, \hat{b}_s^\dagger(\vec{p})\, \frac{\omega}{m} \delta_{rs} \right] \\ &= \sum_{s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega}\, \omega \left[\hat{a}_s^\dagger(\vec{p})\, \hat{a}_s(\vec{p}) - \hat{b}_s(\vec{p})\, \hat{b}_s^\dagger(\vec{p}) \right] \\ &= \sum_{s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega}\, \omega \left[\hat{a}_s^\dagger(\vec{p})\, \hat{a}_s(\vec{p}) + \hat{b}_s^\dagger(\vec{p})\, \hat{b}_s(\vec{p}) - \left\{ \hat{b}_s(\vec{p}), \hat{b}_s^\dagger(\vec{p}) \right\} \right] \end{aligned}$$

ţinând cont de (1.22): $\left\{ \hat{b}_r(\vec{p}), \hat{b}_s^\dagger(\vec{p}\,') \right\} = (2\pi)^3 \frac{\omega}{m}\, \delta^3(\vec{p} - \vec{p}\,')\, \delta_{rs}$

H_D se poate rescrie sub forma:

$$H_D = \sum_{r,s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega}\, \omega \left[\hat{a}_s^\dagger(\vec{p})\, \hat{a}_s(\vec{p}) + \hat{b}_s^\dagger(\vec{p})\, \hat{b}_s(\vec{p}) - (2\pi)^3 \frac{\omega}{m}\, \delta^3(0) \right]$$

Deoarece diversele origini (valoarea zero) ale energiei, nu pot fi măsurate, ultimul termen poate fi ignorat. In acest fel, *Hamiltonian-ul normal ordonat* prin operatorii de creare şi anihilare, este:

$$\boxed{H_D = \sum_{s} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m}{\omega}\, \omega \left[\hat{a}_s^\dagger(\vec{p})\, \hat{a}_s(\vec{p}) + \hat{b}_s^\dagger(\vec{p})\, \hat{b}_s(\vec{p}) \right]} \tag{1.25}$$

Ordonarea normală este definită, pentru câmpurile de spin semi-întreg, ca plasarea tuturor operatorilor $\hat{a}_s^\dagger(\vec{p})$ şi $\hat{b}_s^\dagger(\vec{p})$ (ca operatori de creare) la stânga şi toţi operatorii $\hat{a}_s(\vec{p})$ şi $\hat{b}_s(\vec{p})$ (ca operatori de anihilare) la dreapta.

Diferenţa esenţială între *ordonarea normală* a operatorilor bosonici şi a celor fermionici este apariţia unui semn minus la interschimbarea operatorilor fermionici.

Stările proprii ale operatorului energie

- Să studiem efectul acţiunii operatorilor amplitudinea componentelor dezvoltării Fourier asupra stărilor proprii de energie.

$$\hat{H}_D\,|\hat{\varphi}_n\rangle = E_n\,|\hat{\varphi}_n\rangle \qquad (1.26)$$

Să trecem la calcul, ţinând cont de relaţiile de anticomutare (1.23),

$$H_D\hat{a}_s(\vec{p})\,|\hat{\varphi}_n\rangle = \overbrace{\sum_r\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\left[\hat{a}_r^\dagger(\vec{p}\,')\,\hat{a}_r(\vec{p}\,')+\hat{b}_r^\dagger(\vec{p}\,')\,\hat{b}_r(\vec{p}\,')\right]}^{H_D}\hat{a}_s(\vec{p})\,|\hat{\varphi}_n\rangle$$

trecerea $\hat{a}_s(\vec{p})$ peste o poziţie $\to$ schimbă de semn (1.23),

$$= -\sum_r\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\underbrace{\hat{a}_r^\dagger(\vec{p}\,')\,\hat{a}_s(\vec{p})}\,\hat{a}_r(\vec{p}\,')\,|\hat{\varphi}_n\rangle +$$

trecerea $\hat{a}_s(\vec{p})$ peste două poziţii $\to$ nu schimbă semnul (1.23),

$$+\,\hat{a}_s(\vec{p})\sum_r\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\hat{b}_r^\dagger(\vec{p}\,')\,\hat{b}_r(\vec{p}\,')\,\,|\hat{\varphi}_n\rangle$$

trecerea $\hat{a}_s(\vec{p})$ peste încă o poziţie $\to$ schimbă din nou de semn (1.23),

$$= \sum_r\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\overbrace{\left[\hat{a}_s(\vec{p})\,\hat{a}_r^\dagger(\vec{p}\,') - \underbrace{\left\{\hat{a}_s(\vec{p}),\,\hat{a}_r^\dagger(\vec{p}\,')\right\}}_{(1.23)}\right]}^{\hat{a}_r^\dagger(\vec{p}\,')\hat{a}_s(\vec{p})}\hat{a}_r(\vec{p}\,')\,|\hat{\varphi}_n\rangle +$$

$$+\,\hat{a}_s(\vec{p})\sum_r\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\hat{b}_r^\dagger(\vec{p}\,')\,\hat{b}_r(\vec{p}\,')\,\,|\hat{\varphi}_n\rangle$$

$$= \sum_r\int\underbrace{\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'}_{H_D(1)}\left[\hat{a}_s(\vec{p})\,\underbrace{\hat{a}_r^\dagger(\vec{p}\,')}_{H_D(2)} - (2\pi)^3\frac{\omega}{m}\delta^3(\vec{p}-\vec{p}\,')\,\delta_{rs}\right]\underbrace{\hat{a}_r(\vec{p}\,')}_{H_D(3)}\,|\hat{\varphi}_n\rangle +$$

$$+\,\hat{a}_s(\vec{p})\sum_r\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\underbrace{\hat{b}_r^\dagger(\vec{p}\,')\,\hat{b}_r(\vec{p}\,')}_{H_D(4)}\,|\hat{\varphi}_n\rangle$$

$$= \hat{a}_s(\vec{p})\,H_D\,|\hat{\varphi}_n\rangle - \sum_r\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\left[(2\pi)^3\frac{\omega}{m}\delta^3(\vec{p}-\vec{p}\,')\,\delta_{rs}\right]\hat{a}_r(\vec{p}\,')\,|\hat{\varphi}_n\rangle$$

$$= \hat{a}_s(\vec{p})\,E_n\,|\hat{\varphi}_n\rangle - \omega\,\hat{a}_s(\vec{p})\,|\hat{\varphi}_n\rangle = (E_n-\omega)\hat{a}_s(\vec{p})\,|\hat{\varphi}_n\rangle$$

In concluzie,

$$\boxed{H_D\hat{a}_s(\vec{p})\,|\hat{\varphi}_n\rangle = (E_n-\omega)\,\hat{a}_s(\vec{p})\,|\hat{\varphi}_n\rangle} \qquad (1.27)$$

- Un calcul similar dă un rezultat identic şi pentru operatorul amplitudine Fourier $\hat{b}_s(\vec{p})$.

$$\boxed{H_D\hat{b}_s(\vec{p})\,|\hat{\varphi}_n\rangle = (E_n-\omega)\,\hat{b}_s(\vec{p})\,|\hat{\varphi}_n\rangle} \qquad (1.28)$$

- Deci, operatorii $\hat{a}_s(\vec{p})$ şi $\hat{b}_s(\vec{p})$ sunt operatori de anihilare, sau de coborâre energie, prin care o cuantă de energie $\hbar\omega$ este extrasă din starea asupra căreia acţionează.

- Să trecem să evaluăm şi acţiunea operatorului $\hat{a}_s^\dagger(\vec{p})$ asupra unei stări Dirac de energie $|\hat{\varphi}_n\rangle$.

$$H_D\hat{a}_s^\dagger(\vec{p})\,|\hat{\varphi}_n\rangle = \overbrace{\sum_r \int \frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\left[\hat{a}_r^\dagger(\vec{p}\,')\,\hat{a}_r(\vec{p}\,') + \hat{b}_r^\dagger(\vec{p}\,')\,\hat{b}_r(\vec{p}\,')\right]}^{H_D}\hat{a}_s^\dagger(\vec{p})\,|\hat{\varphi}_n\rangle$$

din (1.23) $\Longrightarrow \hat{a}_r(\vec{p}\,')\hat{a}_s^\dagger(\vec{p}) = -\hat{a}_s^\dagger(\vec{p})\,\hat{a}_r(\vec{p}\,') + (2\pi)^3\frac{\omega}{m}\delta^3(\vec{p}-\vec{p}\,')\,\delta_{rs}$

$$= \sum_r \int \frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\hat{a}_r^\dagger(\vec{p}\,')\left[-\hat{a}_s^\dagger(\vec{p})\,\hat{a}_r(\vec{p}\,') + (2\pi)^3\frac{\omega}{m}\delta^3(\vec{p}-\vec{p}\,')\,\delta_{rs}\right]|\hat{\varphi}_n\rangle +$$

trecerea $\hat{a}_s^\dagger(\vec{p})$ peste două poziţii $\rightarrow$ nu schimbă de semn (1.23),

$$+\,\hat{a}_s^\dagger(\vec{p})\sum_r \int \frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\hat{b}_r^\dagger(\vec{p}\,')\,\hat{b}_r(\vec{p}\,')\,|\hat{\varphi}_n\rangle$$

trecerea $\hat{a}_s^\dagger(\vec{p})$ peste o poziţie $\rightarrow$ schimbă de semn (1.23),

$$= \hat{a}_s^\dagger(\vec{p})\underbrace{\sum_r \int \frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\left[\hat{a}_r^\dagger(\vec{p}\,')\,\hat{a}_r(\vec{p}\,') + \hat{b}_r^\dagger(\vec{p}\,')\,\hat{b}_r(\vec{p}\,')\right]}_{H_D}|\hat{\varphi}_n\rangle +$$

$$+\sum_r \int \frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{m}{\omega'}\,\omega'\,\hat{a}_r^\dagger(\vec{p}\,')\left[(2\pi)^3\frac{\omega}{m}\delta^3(\vec{p}-\vec{p}\,')\delta_{rs}\right]|\hat{\varphi}_n\rangle$$

prin integrarea $d^3\vec{p}\,'$ obţinem,

$$= \hat{a}_s^\dagger(\vec{p})\,H_D\,|\hat{\varphi}_n\rangle + \omega\,\hat{a}_s^\dagger(\vec{p})\,|\hat{\varphi}_n\rangle = (E_n+\omega)\hat{a}_s^\dagger(\vec{p})\,|\hat{\varphi}_n\rangle$$

In concluzie,

$$\boxed{H_D\,\hat{a}_s^\dagger(\vec{p})\,|\hat{\varphi}_n\rangle = (E_n+\omega)\,\hat{a}_s^\dagger(\vec{p})\,|\hat{\varphi}_n\rangle} \qquad (1.29)$$

- Un calcul similar dă un rezultat identic şi pentru operatorul amplitudine Fourier $\hat{b}_s^\dagger(\vec{p})$.

$$\boxed{H_D\,\hat{b}_s^\dagger(\vec{p})\,|\hat{\varphi}_n\rangle = (E_n+\omega)\,\hat{b}_s^\dagger(\vec{p})\,|\hat{\varphi}_n\rangle} \qquad (1.30)$$

- Deci, operatorii $\hat{a}_s^\dagger(\vec{p})$ şi $\hat{b}_s^\dagger(\vec{p})$ sunt operatori de creare, sau de ridicare energie, prin care o cuantă de energie $\hbar\omega$ este adăugată la starea asupra căreia acţionează.