Procese QED în câmpuri laser intense
tema 40

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”Verbum sapienti sat est”

Cuprins

1.1 Cuantificarea câmpului scalar real (Klein-Gordon)

- Pornim cu densitatea de Lagrangian a câmpului Klein-Gordon, scrisă acum cu operatorii de câmp:
$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \hat{\varphi})^2 - m^2 \hat{\varphi}^2\right] \tag{1.1}$$
- Folosind ecuaţia Euler-Lagrange (*ecuaţia de câmp scalar real*) este:
$$\frac{\partial \mathcal{L}}{\partial \hat{\varphi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\varphi})}\right) = 0$$
  cu Lagrangian-ul (1.1) avem $\quad \dfrac{\partial \mathcal{L}}{\partial \hat{\varphi}} = -m^2 \hat{\varphi} \quad ; \quad \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\varphi})} = \partial^\mu \hat{\varphi}$
- Atunci ecuaţia de câmp scalar real cuantic va fi:
$$\left(\partial_\mu \partial^\mu + m^2\right) \hat{\varphi} = 0 \tag{1.2}$$

care este *ecuaţia de câmp cuantic Klein-Gordon* pentru particule scalare de masă m

- Câmpul de impulsuri este $\hat{\pi}(\vec{x}, t) = \partial \mathcal{L} / \partial_0 \hat{\varphi} = \partial^0 \hat{\varphi} = \partial_0 \hat{\varphi}$.
- Densitatea de Hamiltonian de *câmp scalar real* cuantic este:
$$\mathcal{H}_{sr} = \hat{\pi}\, \partial_0 \hat{\varphi} - \mathcal{L} = (\partial_0 \hat{\varphi})^2 - \frac{1}{2}\left[(\partial_\mu \hat{\varphi})^2 - m^2 \hat{\varphi}^2\right] = \frac{1}{2}\Big[\overbrace{\hat{\pi}^2}^{(\partial_0 \hat{\varphi})^2} + (\nabla \hat{\varphi})^2 + m^2 \hat{\varphi}^2\Big] \tag{1.3}$$
- Hamiltonian-ul de câmp scalar real cuantic, este:
$$H_{sr} = \int d^3\vec{x}\ \mathcal{H}_{sr} \tag{1.4}$$
- Soluţia generală de câmp scalar real este o superpoziţie de câmpuri de undă plană, pentru care: $k \cdot x = k_\mu x^\mu, \quad$ cu $\omega \equiv k_0 = \sqrt{\vec{k} \cdot \vec{k} + m^2}$
$$\hat{\varphi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\Big(\hat{a}(\vec{k})\, e^{-i\,k\cdot x} + \hat{a}^\dagger(\vec{k})\, e^{i\,k\cdot x}\Big) \tag{1.5}$$
- Câmpul de impulsuri $\hat{\pi}(\vec{x}, t) = \partial \mathcal{L} / \partial_0 \hat{\varphi} = \partial^0 \hat{\varphi}$ este
$$\hat{\pi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\,(-i\omega)\Big(\hat{a}(\vec{k})\, e^{-i\,k\cdot x} - \hat{a}^\dagger(\vec{k})\, e^{i\,k\cdot x}\Big) \tag{1.6}$$
- Soluţia (1.5) poate fi privită ca o dezvoltare Fourier, ai cărei coeficienţi $\hat{a}(\vec{k})$ şi $\hat{a}^\dagger(\vec{k})$ sunt reciproc conjugaţi Hermitic, astfel încât câmpul scalar $\hat{\varphi}(x)$ este invariant la conjugarea complexă, adică este o mărime reală. De asemenea şi elementul de integrare $\frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}$ este, elementul de spaţiu fazelor, invariant Lorentz.
- Trecerea de la relaţia de comutare între operatorii din mecanica cuantică $\left[\hat{x}, \hat{p}\right] = i\hbar$ la cea între operatorii de câmpuri cuantice (folosind unităţile naturale $c = \hbar = 1$), ne permite exprimarea comutatorilor pentru câmpuri cuantice la acelaşi timp:
$$\begin{cases} \left[\hat{\varphi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)\right] = i\delta^3(\vec{x} - \vec{y}) \\ \left[\hat{\varphi}(\vec{x}, t), \hat{\varphi}(\vec{y}, t)\right] = 0 \\ \left[\hat{\pi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)\right] = 0 \end{cases} \tag{1.7}$$

1.1.1 Amplitudinile Fourier de câmp scalar real (Klein-Gordon)

Determinarea amplitudinilor $\hat{a}(\vec{k})$ şi $\hat{a}^\dagger$.

- Pentru cuantificare câmpurile cuantice $\hat{\varphi}(\vec{x},t)$ şi $\hat{\pi}(\vec{x},t)$, acestea trebuie să fie descrise prin operatori. Descrierea se poate face doar dacă şi coeficienţii Fourier $\hat{a}(\vec{k})$ şi $\hat{a}^\dagger(\vec{k})$ din (1.5) sunt operatori, care vor trebui să satisfacă relaţii particulare de comutare.

  Relaţiile de comutare între operatorii $\hat{a}(\vec{k})$ şi $\hat{a}^\dagger(\vec{k})$ se află plecând de la relaţiile de comutare între câmpurile $\hat{\varphi}$ şi $\hat{\pi}$ (1.7). Pentru aceasta vom proiecta întâi $\hat{\varphi}(\vec{x},t)$ pe componenta de undă plană $e^{i\,k'\cdot x}$:

- Dezvoltarea Fourier (1.5) a câmpului scalar real $\hat{\varphi}(\vec{x},t)$, este:

$$\hat{\varphi}(x)=\int\frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\left(\hat{a}(\vec{k})\,e^{-i\,k\cdot x}+\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot x}\right) \tag{1.5}$$

Proiectăm (1.5) pe componenta armonică $e^{i\,k'\cdot x}$ de funcţii ortogonale:

$$\boxed{\int d^3\vec{x}\,\hat{\varphi}(\vec{x},t)e^{i\,k'\cdot x}}=\int d^3\vec{x}\frac{d^3\vec{k}}{(2\pi)^3 2\omega}\left[\hat{a}(\vec{k})e^{i(k'-k)\cdot x}+\hat{a}^\dagger(\vec{p})e^{i\,(k'+k)\cdot x}\right]$$

$$=\int\underbrace{d^3\vec{x}}\,\frac{d^3\vec{k}}{\underbrace{(2\pi)^3}\,2\omega}\left[\hat{a}(\vec{k})\,e^{i\,(\omega'-\omega)t}\,\underbrace{e^{-i\,(\vec{k}\,'-\vec{k})\cdot\vec{x}}}+\right.$$

$$\left.+\hat{a}^\dagger(\vec{k})\,e^{i\,(\omega'+\omega)t}\,\underbrace{e^{-i\,(\vec{k}\,'+\vec{k})\cdot\vec{x}}}\right]$$

Pentru integrala după $d^3\vec{x}$, folosim funcţia δ:

$$\int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}-\vec{k}')\cdot\vec{x}}=\delta^3(\vec{k}\,'-\vec{k})\quad;\quad\int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}+\vec{k}')\cdot\vec{x}}=\delta^3(\vec{k}'+\vec{k})$$

Atunci, efectuând integrala după $d^3\vec{x}$, apoi după $d^3\vec{k}$, folosind relaţia $\omega\equiv k_0=\sqrt{\vec{k}\cdot\vec{k}+m^2}$, obţinem:

$$\boxed{\int d^3\vec{x}\;\hat{\varphi}(\vec{x},t)\,e^{i\,k'\cdot x}}=\int\frac{d^3\vec{k}}{2\omega}\left[\hat{a}(\vec{k})\,e^{i\,(\omega'-\omega)t}\,\delta^3(\vec{k}\,'-\vec{k})+\hat{a}^\dagger(\vec{k})e^{i\,(\omega'+\omega)t}\,\delta^3(\vec{k}\,'+\vec{k})\right]$$

$$=\frac{1}{2\omega'}\left[a(\vec{k}\,')+\hat{a}^\dagger(-\vec{k}\,')\,e^{i\,2\,\omega' t}\right]$$

Acum, dacă am făcut şi integrarea după $\vec{k}$, putem elimina semnul prim ataşat componentelor $k=(\omega,\vec{k})$, astfel că rezultatul final al integrării este:

$$\boxed{\int d^3\vec{x}\;\hat{\varphi}(\vec{x},t)\,e^{i\,k\cdot x}}=\frac{1}{2\omega}\left[a(\vec{k})+\hat{a}^\dagger(-\vec{k})\,e^{i\,2\,\omega t}\right] \tag{1.8}$$

- Vom repeta aceleaşi calcule pentru câmpul $\hat{\pi}(\vec{x},t)$. Proiectând $\hat{\pi}(\vec{x},t)$ (1.6) pe aceeaşi componentă armonică $e^{i\,k'\cdot\,x}$ de funcţii ortogonale, calculăm integrala $d^3\vec{x}$:

$$\boxed{\int d^3\vec{x}\,\hat{\pi}(\vec{x},t)e^{i\,k'\cdot\,x}} = \int d^3\vec{x}\frac{d^3\vec{k}}{(2\pi)^3 2\omega}(-i\omega)\Big[\hat{a}(\vec{k})e^{i(k'-k)\,\cdot\,x} - \hat{a}^\dagger(\vec{k})e^{i\,(k'+k)\cdot\,x}\Big]$$

$$= \int \underbrace{d^3\vec{x}}\,\frac{d^3\vec{k}}{\underbrace{(2\pi)^3}\,2\omega}(-i\omega)\Big[\hat{a}(\vec{k})\,e^{i\,(\omega'-\omega)t}\;\underbrace{e^{-i\,(\vec{k}\,'-\vec{k})\,\cdot\,\vec{x}}} -$$

$$-\hat{a}^\dagger(\vec{k})\,e^{i\,(\omega'+\omega)t}\;\underbrace{e^{-i\,(\vec{k}\,'+\vec{k})\cdot\,\vec{x}}}\Big]$$

Pentru integrala după x, folosim funcţia δ:

$$\int \frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}-\vec{k}\,')\cdot\,\vec{x}} = \delta^3(\vec{k}\,'k - \vec{k}) \quad ; \quad \int \frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}+\vec{k}\,')\cdot\,\vec{x}} = \delta^3(\vec{k}\,' + \vec{k})$$

Atunci, efectuând integrala după $d^3\vec{x}$, apoi după $d^3\vec{k}$, folosind relaţia
$\omega \equiv k_0 = \sqrt{\vec{k}\cdot\vec{k} + m^2}$, obţinem,

$$\boxed{\int d^3\vec{x}\;\hat{\pi}(\vec{x},t)\,e^{i\,k'\cdot\,x}} = \int \frac{d^3\vec{k}}{2\omega}(-i\,\omega)\Big[\hat{a}(\vec{k})\,e^{i\,(\omega\,'-\omega)t}\,\delta^3(\vec{k}\,'-\vec{k}) -$$

$$-\hat{a}^\dagger(\vec{k})e^{i\,(\omega\,'+\omega)t}\,\delta^3(\vec{k}\,'+\vec{k})\Big]$$

$$= -\frac{i}{2}\Big[\hat{a}(\vec{k}\,') - \hat{a}^\dagger(-\vec{k}\,')\,e^{i\,2\,\omega\,'t}\Big]$$

Acum, dacă am făcut integrarea după $\vec{k}$, putem elimina semnul prim ataşat componentelor $k = (\omega,\vec{k})$, astfel că rezultatul integrării este:

$$\boxed{\int d^3\vec{x}\;\hat{\pi}(\vec{x},t)\,e^{i\,k\,\cdot\,x}} = -\frac{i}{2}\Big[\hat{a}(\vec{k}) - \hat{a}^\dagger(-\vec{k})\,e^{i\,2\,\omega t}\Big] \tag{1.9}$$

- Folosind cele două relaţii obţinute (1.8) şi (1.9) putem exprima amplitudinile $\hat{a}(\vec{k})$ şi $\hat{a}^\dagger(\vec{k})$ ale dezvoltării Fourier (1.5) a câmpului cuantic scalar real $\hat{\varphi}(x)$, prin câmpurile $\hat{\varphi}$ şi $\hat{\pi}$:

$$\boxed{\hat{a}(\vec{k}) = \int d^3\vec{x}\;e^{i\,k\,\cdot\,x}\Big[\omega\,\hat{\varphi}(\vec{x},t) + i\,\hat{\pi}(\vec{x},t)\Big]}$$

$$\boxed{\hat{a}^\dagger(\vec{k}) = \int d^3\vec{x}\;e^{-i\,k\,\cdot\,x}\Big[\omega\,\hat{\varphi}(\vec{x},t) - i\,\hat{\pi}(\vec{x},t)\Big]}$$

- Operatorii de câmp $\hat{a}(\vec{k})$ şi $\hat{a}^\dagger(\vec{k})$, sunt operatori de coborâre (anihilare) şi ridicare (creare) de stări de energie ale câmpului scalar real (vezi mai departe), similar cu aceiaşi operatori găsiţi pentru oscilatorul armonic cuantic.

  Pentru a face cuantificarea câmpului scalar real $\hat{\varphi}(\vec{x},t)$ va trebui să determinăm relaţiile de comutare pentru aceşti operatori.

1.1.2 Relaţii de comutare operatori de câmp scalar real (Klein-Gordon)

- Să evaluăm relaţiile de comutare $\boxed{\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right]}$ folosind expresiile amplitudinilor Fourier (1.5) $\hat{a}(\vec{k})$ şi $\hat{a}^\dagger(\vec{k})$ prin operatorii de câmp scalar real la acelaşi timp.

$$\boxed{\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(k\cdot x - k'\cdot x')}$$

$$\Big[\overbrace{\Big(\omega\,\hat{\varphi}(\vec{x},t) + i\,\hat{\pi}(\vec{x},t)\Big)}^{\hat{a}(\vec{k})}\overbrace{\Big(\omega\,\hat{\varphi}(\vec{x}\,',t) - i\,\hat{\pi}(\vec{x}\,',t)\Big)}^{\hat{a}^\dagger(\vec{k}\,')} -$$

$$- \overbrace{\Big(\omega\,\hat{\varphi}(\vec{x}\,',t) - i\,\hat{\pi}(\vec{x}\,',t)\Big)}^{\hat{a}^\dagger(\vec{k}\,')}\overbrace{\Big(\omega\,\hat{\varphi}(\vec{x},t) + i\,\hat{\pi}(\vec{x},t)\Big)}^{\hat{a}(\vec{k})}\Big]$$

- Folosind comutatorii din (1.7) $\left[\hat{\varphi}(\vec{x},t), \hat{\varphi}(\vec{y},t)\right] = 0$ şi $\left[\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{y},t)\right] = 0$

$$\boxed{\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(k\cdot x - k'\cdot x')} \times$$

$$\times \left[i\,\omega\Big(\underbrace{\hat{\pi}(\vec{x},t)\hat{\varphi}(\vec{x}\,',t) - \hat{\varphi}(\vec{x}\,'t)\hat{\pi}(\vec{x},t)}\Big) - i\,\omega\Big(\underbrace{\hat{\varphi}(\vec{x},t)\hat{\pi}(\vec{x}\,',t) - \hat{\pi}(\vec{x}\,',t)\hat{\varphi}(\vec{x},t)}\Big)\right]$$

$$\boxed{\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(k\cdot x - k'\cdot x')}\left(i\,\omega\,\underbrace{\overbrace{\left[\hat{\pi}(\vec{x},t), \varphi(\vec{x}\,',t)\right]}}_{-i\delta^3(\vec{x}-\vec{x}\,')} - i\,\omega\,\underbrace{\overbrace{\left[\hat{\varphi}(\vec{x},t), \hat{\pi}(\vec{x}\,',t)\right]}}_{i\delta^3(\vec{x}-\vec{x}\,')}\right)$$

Folosind primul comutator (1.7) $\left[\hat{\varphi}(\vec{x},t), \hat{\pi}(\vec{y},t)\right] = i\delta^3(\vec{x}-\vec{y})$ şi integrând $d^3\vec{x}\,'$,

$$\boxed{\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(k\cdot x - k'\cdot x')}\, 2\,\omega\,\delta^3(\vec{x}-\vec{x}\,') = 2\omega\int d^3\vec{x}\; e^{i(k-k')\cdot x}$$

$$= 2\omega\underbrace{\int d^3\vec{x}}\; e^{i\,(\omega-\omega')t}\;\underbrace{e^{-i\,(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta^3(\vec{k}\,'-\vec{k})} = (2\pi)^3\, 2\omega\,\underbrace{e^{i\,(\omega-\omega')t}}_{\omega=\omega'}\,\delta^3(\vec{k}\,'-\vec{k})$$

$$= (2\pi)^3\, 2\omega\,\delta^3(\vec{k}-\vec{k}\,')$$

- Final, comutatorul căutat este: $$\boxed{\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right] = (2\pi)^3\, 2\omega\,\delta^3(\vec{k}-\vec{k}\,')} \qquad (1.10)$$

- Imediat se poate arăta că $\hat{a}(\vec{k})$ comută cu el însuşi. La fel şi $\hat{a}^\dagger(\vec{k})$ comută cu el însăşi.

$$\boxed{\left[\hat{a}(\vec{k}), \hat{a}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\, e^{i\,(k\cdot x + k'\cdot x')} \left(i\,\omega \underbrace{\left[\hat{\varphi}(\vec{x},t), \hat{\pi}(\vec{x}\,',t)\right]}_{i\,\delta^3(\vec{x}-\vec{x}\,')} + i\,\omega \underbrace{\left[\hat{\pi}(\vec{x},t), \hat{\varphi}(\vec{x}\,',t)\right]}_{-i\,\delta^3(\vec{x}-\vec{x}\,')} \right)$$

$$= \int d^3\vec{x}\; d^3\vec{x}\,'\, e^{i\,(k\cdot x + k'\cdot x')} \Big(-\,\omega\,\delta^3(\vec{x}-\vec{x}\,') + \omega\,\delta^3(\vec{x}-\vec{x}\,') \Big) = 0$$

Deci, comutatorul căutat este:

$$\boxed{\left[\hat{a}(\vec{k}), \hat{a}(\vec{k}\,')\right] = 0} \tag{1.11}$$

La fel,

$$\boxed{\left[\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\, e^{-i\,(k\cdot x + k'\cdot x')} \left(-i\,\omega \underbrace{\left[\hat{\varphi}(\vec{x},t), \hat{\pi}(\vec{x}\,',t)\right]}_{i\,\delta^3(\vec{x}-\vec{x}\,')} - i\,\omega \underbrace{\left[\hat{\pi}(\vec{x},t), \hat{\varphi}(\vec{x}\,',t)\right]}_{-i\,\delta^3(\vec{x}-\vec{x}\,')} \right)$$

$$= \int d^3\vec{x}\; d^3\vec{x}\,'\, e^{-i\,(k\cdot x + k'\cdot x')} \Big(\omega\,\delta^3(\vec{x}-\vec{x}\,') - \omega\,\delta^3(\vec{x}-\vec{x}\,') \Big) = 0$$

Deci, comutatorul căutat este:

$$\boxed{\left[\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right] = 0} \tag{1.12}$$

- In rezumat, setul complet de relaţii de comutare pentru operatorii coeficienţi Fourier ai dezvoltării câmpului scalar real, este:

$$\boxed{\begin{aligned} \left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right] &= (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}\,') \\ \left[\hat{a}(\vec{k}), \hat{a}(\vec{k}\,')\right] &= 0 \\ \left[\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right] &= 0 \end{aligned}} \tag{1.13}$$

1.1.3 Hamiltonian de câmp scalar real (Klein-Gordon)

- Cu ajutorul relaţiilor de comutare stabilite mai înainte, vom trece la cuantificarea câmpului scalar real. La început vom căuta expresia Hamiltonian-ul acestui câmp şi apoi vom determina stările proprii ale Hamiltonian-ului.
- Folosind (1.3), Hamiltonian-ul de câmp cuantic scalar real, este:

$$\hat{H}_{sr}=\int d^3\vec{x}\ \frac{1}{2}\Big(\hat{\pi}^2(\vec{x})+(\nabla\hat{\varphi})^2+m^2\hat{\varphi}^2\Big) \tag{1.14}$$

unde
$$\begin{cases}\hat{\varphi}(\vec{x})=\displaystyle\int\frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\Big(\hat{a}(\vec{k})\,e^{-i\,k\cdot x}+\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot x}\Big)\\ \hat{\pi}(\vec{x})=\displaystyle\int\frac{d^3\vec{k}}{(2\pi)^3\,2\omega}(-i\omega)\Big(\hat{a}(\vec{k})\,e^{-i\,k\cdot x}-\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot x}\Big)\\ \nabla\hat{\varphi}(\vec{x})=\displaystyle\int\frac{d^3\vec{k}}{(2\pi)^3\,2\omega}(i\vec{k})\Big(\hat{a}(\vec{k})\,e^{-i\,k\cdot x}-\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot\,x}\Big)\end{cases}$$

atunci,

$$\begin{aligned}H_{sr}=&\int d^3\vec{x}\ \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\ \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\times\\ &\times\frac{1}{2}\Big\{-\omega\,\omega'\Big[\hat{a}(\vec{k})\,e^{-i\,k\cdot x}-\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot x}\Big]\Big[\hat{a}(\vec{k}\,')\,e^{-i\,k'\cdot x}-\hat{a}^\dagger(\vec{k}\,')\,e^{i\,k'\cdot x}\Big]-\\ &\quad-\vec{k}\cdot\vec{k}\,'\Big[\hat{a}(\vec{k})\,e^{-i\,k\cdot x}-\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot x}\Big]\Big[\hat{a}(\vec{k}\,')\,e^{-i\,k'\cdot x}-\hat{a}^\dagger(\vec{k}\,')\,e^{i\,k'\cdot x}\Big]+\\ &\quad+m^2\Big[\hat{a}(\vec{k})\,e^{-i\,k\cdot x}+\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot x}\Big]\Big[\hat{a}(\vec{k}\,')\,e^{-i\,k'\cdot x}+\hat{a}^\dagger(\vec{k}\,')\,e^{i\,k'\cdot x}\Big]\Big\}\end{aligned}$$

$$\begin{aligned}H_{sr}=&\int d^3\vec{x}\ \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\ \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\times\\ &\times\frac{1}{2}\Big\{\Big(-\omega\,\omega'-\vec{k}\cdot\vec{k}\,'\Big)\Big[\hat{a}(\vec{k})\,e^{-i\,k\cdot x}-\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot x}\Big]\Big[\hat{a}(\vec{k}\,')\,e^{-i\,k'\cdot x}-\hat{a}^\dagger(\vec{k}\,')\,e^{i\,k'\cdot x}\Big]+\\ &\quad+m^2\Big[\hat{a}(\vec{k})\,e^{-i\,k\cdot x}+\hat{a}^\dagger(\vec{k})\,e^{i\,k\cdot x}\Big]\Big[\hat{a}(\vec{k}\,')\,e^{-i\,k'\cdot x}+\hat{a}^\dagger(\vec{k}\,')\,e^{i\,k'\cdot x}\Big]\Big\}\end{aligned}$$

integralele după $d^3\vec{x}$,

$$\begin{aligned}\int d^3\vec{x}\ e^{\pm i\,(k-k')\cdot x}&=\int \underbrace{d^3\vec{x}}\ e^{\pm i\,(\omega-\omega')\,t}\ \underbrace{e^{\mp i\,(\vec{k}-\vec{k}\,')\cdot\vec{x}}}\\ &=e^{\pm i\,(\omega-\omega')\,t}\ \overbrace{(2\pi)^3\,\delta^3(\vec{k}-\vec{k}\,')}=(2\pi)^3\delta^3(\vec{k}-\vec{k}\,')\end{aligned}$$

$$\begin{aligned}\int d^3\vec{x}\ e^{\pm i\,(k+k')\cdot x}&=\int \underbrace{d^3\vec{x}}\ e^{\pm i\,(\omega+\omega')\,t}\ \underbrace{e^{\mp i\,(\vec{k}+\vec{k}\,')\cdot\vec{x}}}\\ &=e^{\pm i\,(\omega+\omega')\,t}\ \overbrace{(2\pi)^3\,\delta^3(\vec{k}+\vec{k}\,')}=e^{\pm 2i\,\omega t}\,(2\pi)^3\,\delta^3(\vec{k}+\vec{k}\,')\end{aligned}$$

$$H_{sr} = \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\, \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\, \frac{1}{2} \Bigg\{ \underbrace{\Big(-\omega\,\omega' - \vec{k}\cdot\vec{k}\,'\Big)}\, \times$$

$$\times \Big[\hat{a}(\vec{k})\hat{a}(\vec{k}\,')(2\pi)^3\, e^{-2i\,\omega t}\delta^3(\vec{k}+\vec{k}\,') - \underbrace{\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}\,')(2\pi)^3\,\delta^3(\vec{k}-\vec{k}\,')} -$$

$$- \underbrace{\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}\,')(2\pi)^3\,\delta^3(\vec{k}-\vec{k}\,')} + \hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k}\,')(2\pi)^3\, e^{2i\,\omega t}\delta^3(\vec{k}+\vec{k}\,') \Big] +$$

$$\underbrace{+m^2} \times$$

$$\times \Big[\hat{a}(\vec{k})\hat{a}(\vec{k}\,')(2\pi)^3\, e^{-2i\,\omega t}\delta^3(\vec{k}+\vec{k}\,') + \underbrace{\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}\,')(2\pi)^3\,\delta^3(\vec{k}-\vec{k}\,')} +$$

$$+ \underbrace{\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}\,')(2\pi)^3\,\delta^3(\vec{k}-\vec{k}\,')} + \hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k}\,')(2\pi)^3\, e^{2i\,\omega t}\delta^3(\vec{k}+\vec{k}\,') \Big] \Bigg\}$$

- Folosim proprietăţile funcţiei δ şi relaţia $\omega \equiv k_0 = \sqrt{\vec{k}\cdot\vec{k}+m^2}$, astfel pentru termenii cu
  $\delta^3(\vec{k}+\vec{k}\,') \Longrightarrow \vec{k} = -\vec{k}\,'$ şi $\omega\omega' = \omega^2 \Longrightarrow -\omega\omega' - \vec{k}\cdot\vec{k}\,' + m^2 = -\omega^2 + k^2 + m^2 = 0$
  Deci, termenii cu $\delta^3(\vec{k}+\vec{k}\,')$ se anulează reciproc, iar în H_{rs} rămâne:

$$H_{sr} = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega}\, \frac{d^3\vec{k}\,'}{(2\pi)^3 2\omega'} (2\pi)^3 \delta^3(\vec{k}-\vec{k}\,') \Bigg\{ \Big(\omega\,\omega' + \vec{k}\cdot\vec{k}\,' + m^2\Big) \Big[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}\,') + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}\,')\Big] \Bigg\}$$

prin integrarea $d^3\vec{k}\,'$ cu $\delta^3(\vec{k}-\vec{k}\,')$, avem $\vec{k} = \vec{k}\,'$ şi $\omega = \omega'$, obţinem:

$$H_{sr} = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\, \frac{(2\pi)^3}{(2\pi)^3\, 2\omega} \Bigg\{ \underbrace{\Big(\omega^2 + \vec{k}\cdot\vec{k} + m^2\Big)}_{2\omega^2} \Big[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\Big] \Bigg\}$$

$$= \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\, \frac{(2\pi)^3}{(2\pi)^3\, 2\omega} 2\omega^2 \Big[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\Big]$$

deci, Hamiltonian-ul de câmp scalar real este:

$$H_{sr} = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega} \omega \Big[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\Big]$$

- Folosind relaţiie de comutare (1.13) $\Big[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\Big] = (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}\,')$
  putem asigura ordonarea normală $\hat{a}^\dagger(\vec{k})\,\hat{a}(\vec{k})$ a operatorilor,

$$H_{sr} = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega} \omega \Big[2\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + (2\pi)^3 2\omega\, \delta^3(0)\Big]$$

Deoarece diversele origini (valoarea zero) ale energiei, nu pot fi măsurate, ultimul termen poate fi neglijat. In acest fel, *Hamiltonian-ul normal ordonat de câmp scalar real* (Klein-Gordon), exprimat prin operatorii de creare şi anihilare, este:

$$\boxed{H_{sr} = \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\, \omega\, \hat{a}^\dagger(\vec{k})\, \hat{a}(\vec{k})} \tag{1.15}$$

Creare şi anihilare cuante de energie de câmp scalar real

- Să studiem efectul acţiunii operatorilor amplitudinea componentelor dezvoltării Fourier asupra stărilor proprii de energie ale câmpului cuantic scalar real.

$$\hat{H}_{sr}\, |\hat{\varphi}_n\rangle = E_n\, |\hat{\varphi}_n\rangle \tag{1.16}$$

Să trecem la calcul, ţinând cont de relaţiile de anticomutare (1.13),

$$\begin{aligned}
H_{sr}\; \underbrace{\hat{a}(\vec{k})\Big|\varphi_n\Big\rangle} &= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\, \omega'\, \hat{a}^\dagger(\vec{k}\,')\, \hat{a}(\vec{k}\,')\, \hat{a}(\vec{k})\Big|\varphi_n\Big\rangle \\
&= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\, \omega'\, \hat{a}^\dagger(\vec{k}\,')\, \hat{a}(\vec{k})\, \hat{a}(\vec{k}\,')\Big|\varphi_n\Big\rangle \\
&= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\, \omega' \left(\hat{a}(\vec{k})\, \hat{a}^\dagger(\vec{k}\,') - \Big[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\Big] \right) \hat{a}(\vec{k}\,')\Big|\varphi_n\Big\rangle \\
&= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\, \omega' \left(\hat{a}(\vec{k})\, \hat{a}^\dagger(\vec{k}\,') - (2\pi)^3\, 2\omega\, \delta^3(\vec{k} - \vec{k}\,') \right) \hat{a}(\vec{k}\,')\Big|\varphi_n\Big\rangle \\
&= \hat{a}(\vec{k}) \underbrace{\int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\, \omega' \hat{a}^\dagger(\vec{k}\,')\, \hat{a}(\vec{k}\,')}_{H_{sr}}\Big|\varphi_n\Big\rangle - \omega\; \underbrace{\hat{a}(\vec{k})\Big|\varphi_n\Big\rangle} \\
&= \hat{a}(\vec{k}) H_{sr}\Big|\varphi_n\Big\rangle - \omega\; \underbrace{\hat{a}(\vec{k})\Big|\varphi_n\Big\rangle} = \hat{a}(\vec{k}) E_n\Big|\varphi_n\Big\rangle - \omega\; \underbrace{\hat{a}(\vec{k})\Big|\varphi_n\Big\rangle} \\
&= (E_n - \omega)\; \underbrace{\hat{a}(\vec{k})\Big|\varphi_n\Big\rangle}
\end{aligned}$$

In concluzie,

$$\boxed{H_{sr}\; \underbrace{\hat{a}(\vec{k})\Big|\varphi_n\Big\rangle}_{|\varphi_{n-1}\rangle} = (E_n - \omega)\; \underbrace{\hat{a}(\vec{k})\Big|\varphi_n\Big\rangle}_{|\varphi_{n-1}\rangle}} \tag{1.17}$$

- Deci, operatorul $\hat{a}(\vec{k})$ este operator de coborâre sau anihilare a stării $\Big|\varphi_n\Big\rangle$ asupra căreia acţionează, conducând la starea de energie $E_n - \hbar\omega$, adică cea cu o cuantă de energie $\hbar\omega$ mai mică..

- Să trecem acum să evaluăm şi acţiunea operatorului $\hat{a}^\dagger(\vec{k})$ asupra unei stări de câmp scalar real de energie $\Big|\varphi_n\Big\rangle$

$$
\begin{aligned}
H_{sr}\ \underbrace{\hat{a}^\dagger(\vec{k})\Big|\varphi_n\Big\rangle} &= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\,\omega'\,\hat{a}^\dagger(\vec{k}\,')\,\hat{a}(\vec{k}\,')\,\hat{a}^\dagger(\vec{k})\Big|\varphi_n\Big\rangle \\
&= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\,\omega'\,\hat{a}^\dagger(\vec{k}\,')\left(\hat{a}^\dagger(\vec{k})\,\hat{a}(\vec{k}\,') - \left[\hat{a}^\dagger(\vec{k}),\hat{a}(\vec{k}\,')\right]\right)\Big|\varphi_n\Big\rangle \\
&= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\,\omega'\,\hat{a}^\dagger(\vec{k}\,')\left(\hat{a}^\dagger(\vec{k})\,\hat{a}(\vec{k}\,') + \left[\hat{a}(\vec{k}\,'),\hat{a}^\dagger(\vec{k})\right]\right)\Big|\varphi_n\Big\rangle \\
&= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\,\omega'\,\hat{a}^\dagger(\vec{k}\,')\left(\hat{a}^\dagger(\vec{k})\,\hat{a}(\vec{k}\,') + (2\pi)^3\, 2\omega\,\delta^3(\vec{k}-\vec{k}\,')\right)\Big|\varphi_n\Big\rangle \\
&= \int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\,\omega'\hat{a}^\dagger(\vec{k}\,')\,\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}\,')\Big|\varphi_n\Big\rangle + \omega\ \underbrace{\hat{a}^\dagger(\vec{k})\Big|\varphi_n\Big\rangle} \\
&= \hat{a}^\dagger(\vec{k})\ \underbrace{\int \frac{d^3\vec{k}\,'}{(2\pi)^3\, 2\omega'}\,\omega'\hat{a}^\dagger(\vec{k}\,')\,\hat{a}(\vec{k}\,')}_{H_{sr}}\Big|\varphi_n\Big\rangle + \omega\ \underbrace{\hat{a}^\dagger(\vec{k})\Big|\varphi_n\Big\rangle} \\
&= \hat{a}^\dagger(\vec{k})H_{sr}\Big|\varphi_n\Big\rangle + \omega\ \underbrace{\hat{a}^\dagger(\vec{k})\Big|\varphi_n\Big\rangle} = \hat{a}^\dagger(\vec{k})E_n\Big|\varphi_n\Big\rangle + \omega\ \underbrace{\hat{a}^\dagger(\vec{k})\Big|\varphi_n\Big\rangle} \\
&= (E_n+\omega)\ \underbrace{\hat{a}^\dagger(\vec{k})\Big|\varphi_n\Big\rangle}
\end{aligned}
$$

In concluzie,

$$
\boxed{H_{sr}\ \underbrace{\hat{a}^\dagger(\vec{k})\Big|\varphi_n\Big\rangle}_{|\varphi_{n+1}\rangle} = (E_n+\omega)\ \underbrace{\hat{a}(\vec{k})\Big|\varphi_n\Big\rangle}_{|\varphi_{n+1}\rangle}} \tag{1.18}
$$

- Deci, operatorul $\hat{a}^\dagger(\vec{k})$ este operator de ridicare sau creare a stării $\big|\varphi_{n+1}\big\rangle$, cu o cuantă de energie $\hbar\omega$ mai mare, prin acţiunea asupra stării $\big|\varphi_n\big\rangle$,