Procese QED în câmpuri laser intense
tema 41

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”Verbum sapienti sat est”

Cuprins

1.1 Cuantificarea câmpului electromagnetic (Maxwell)

- Cuantificarea câmpului electromagnetic, la fel ca a oricărui câmp clasic, se face prin definirea relaţiilor de comutare între operatorii de câmp, aici $\hat{A}^\mu$, şi cei de impuls canonic conjugat $\hat{\pi}^\mu$.

- Densitatea de Lagrangian a câmpului electromagnetic liber (cu $j_\mu = 0$), scrisă acum cu operatorii de câmp:

$$\mathcal{L} = \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} \tag{1.1}$$

unde operatorul tensor electromagnetic $\hat{F}_{\mu\nu}$ este:

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$$
$$\hat{F}^{\mu\nu} = \partial^\mu \hat{A}^\nu - \partial^\nu \hat{A}^\mu$$

Expresia explicită $\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}$ este:

$$\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} = 2\left(\partial_1\hat{A}_2 - \partial_2\hat{A}_1\right)^2 + 2\left(\partial_2\hat{A}_3 - \partial_3\hat{A}_2\right)^2 + 2\left(\partial_3\hat{A}_1 - \partial_1\hat{A}_3\right)^2 - \\ -2\left(\partial_0\hat{A}_1 - \partial_1\hat{A}_0\right)^2 - 2\left(\partial_0\hat{A}_2 - \partial_2\hat{A}_0\right)^2 - 2\left(\partial_0\hat{A}_3 - \partial_3\hat{A}_0\right)^2$$

- Astfel, câmpul de impulsuri, ca derivata Lagrangian-ului în raport cu derivata temporală a câmpului $\hat{A}_\mu$, este:

$$\hat{\pi}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 \hat{A}_\mu)} = \frac{1}{4}\left[-4\left(\partial_0\hat{A}_\mu - \partial_\mu\hat{A}_0\right)\right] = -\hat{F}_{0\mu} = \hat{F}_{\mu 0} \tag{1.2}$$

Câmpul de impulsuri se identifică cu componentele electrice ale câmpului electromagnetic. Intr-adevăr, conform

$$\boxed{\begin{aligned}\hat{\pi}^0 &= 0\\ \hat{\pi}^i &= \hat{E}^i\end{aligned}} \tag{1.3}$$

- Incercăm la început să facem cuantificarea câmpului electromagnetic căutând relaţiile de comutare între operatorii de câmp $\hat{A}^\mu$ şi cei de câmp de impuls canonic conjugat $\hat{\pi}^\mu$, la fel ca în cazul câmpului scalar real.

- Trecerea de la relaţia de comutare între operatorii din mecanica cuantică $\left[\hat{x}, \hat{p}\right] = i\hbar$ la cea între operatorii de câmpuri cuantice (în unităţi cu $c = \hbar = 1$), ar duce la comutatorii:

$$\left[\hat{A}^\mu(\vec{x}, t), \hat{\pi}^\mu(\vec{y}, t)\right] = i\delta^3(\vec{x} - \vec{y}) \tag{1.4}$$

Dar dacă $\hat{\pi}^0 = 0$, relaţia de comutare între $\hat{A}^0 \equiv \hat{\phi}$ şi valoarea 0 nu poate fi diferită de zero. Această înseamnă şi violarea invarianţei la transformarea de calibrare.

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi \tag{1.5}$$

Putem alege în acest caz o calibrare particulară, *Calibrarea Coulomb*, care face alegerea câmpului vectorial $\vec{A}$ ca unul cu oscilaţii transversale faţă de direcţia de propagare.

$$\nabla \cdot \vec{A} = 0 \quad \text{şi} \quad A^0 \equiv \phi = 0 \tag{1.6}$$

- Acum putem încerca să scriem ca de obcei, relaţiile de comutare între operatorii de câmp $\hat{A}_i(\vec{x},t)$ şi de impuls canonic $\hat{\pi}^j(\vec{x}\,',t) \equiv \hat{E}^j(\vec{x}\,',t)$, luaţi la acelaşi timp,

$$\left[\hat{A}_i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] = i\,\delta_i^j\,\delta^3(\vec{x}-\vec{x}\,') \tag{1.7}$$

- Să vedem acum ce devine (1.7), după ce câmpul şi coeficienţii dezvoltării Fourier au devenit operatori.

- Dacă păstrăm relaţiile $\nabla\cdot\vec{A}=0$ şi $\nabla\cdot\vec{E}=0$ valabile şi pentru operatori, şi aplicăm operatorul ∇ asupra (1.7), obţinem o contradicţie:

$$0 = \left[\overbrace{\nabla\cdot\hat{\vec{A}}(\vec{x},t)}^{=0}\,,\;\overbrace{\nabla\cdot\hat{\vec{E}}(\vec{x}\,',t)}^{=0}\right] = i\,\nabla^2\delta^3(\vec{x}-\vec{x}\,') \neq 0$$

- Pentru rezolvarea acestei inadvertenţe putem scrie comutatorul (1.7) sub forma:

$$\left[\hat{A}_i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] = i\,\left(\delta_i^j - \frac{\partial_i\partial^j}{\nabla^2}\right)\delta^3(\vec{x}-\vec{x}\,') \tag{1.8}$$

- Exprimând $\delta^3(\vec{x}-\vec{x}\,')=\int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\,\vec{k}\,\cdot\,(\vec{x}-\vec{x}\,')}$ şi efectuând $\partial_i\partial^j$ şi ∇, obţinem:

$$\left[\hat{A}_i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] = i\int\frac{d^3\vec{k}}{(2\pi)^3}\left(\delta_i^j - \frac{k_ik^j}{k^2}\right)e^{i\,\vec{k}\,\cdot\,(\vec{x}-\vec{x}\,')} \tag{1.9}$$

- In concluzie, setul complet de relaţii de comutare pentru operatorii de câmp electromagnetic cuantic, luaţi la acelaşi timp, va fi,

$$\boxed{\begin{aligned}\left[\hat{A}_i(\vec{x},t), \hat{A}_j(\vec{x}\,',t)\right] &= 0\\ \left[\hat{E}^i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] &= 0\\ \left[\hat{A}_i(\vec{x},t), \hat{E}_j(\vec{x}\,',t)\right] &= i\int\frac{d^3\vec{k}}{(2\pi)^3}\left(\delta_i^j - \frac{k_ik^j}{k^2}\right)e^{i\,\vec{k}(\vec{x}-\vec{x}\,')}\end{aligned}} \tag{1.10}$$

1.1.1 Ecuaţia de câmp electromagnetic în calibrarea Coulomb

- Ecuaţia de câmp electromagnetic liber (cu $j^\nu = 0$), în calibrarea Coulomb, este:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0 \\ \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) &= 0 \end{aligned} \tag{1.11}$$

sau

$$\partial_\mu \left(\partial^\mu A^0 - \partial^0 A^\mu\right) = 0 \quad ; \quad \partial_\mu \left(\partial^\mu \vec{A} + \nabla A^\mu\right) = 0$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\partial_\mu\partial^\mu A^0 - \partial_\mu\partial^0 A^\mu = \quad ; \quad \partial_\mu\partial^\mu \vec{A} + \partial_\mu \nabla A^\mu =$$

$$= \overbrace{\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi} - \overbrace{\left(\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\nabla\cdot\vec{A}\right)} = 0 \quad ; \quad = \overbrace{\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A}} + \overbrace{\frac{\partial}{\partial t}\nabla\phi + \nabla(\nabla\cdot\vec{A})} = 0$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\underbrace{\nabla^2\phi}_{=0} + \frac{\partial}{\partial t}\underbrace{\nabla\cdot\vec{A}}_{=0} = 0 \quad ; \quad \frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} + \frac{\partial}{\partial t}\underbrace{\nabla\phi}_{=0} + \nabla(\underbrace{\nabla\cdot\vec{A}}_{=0}) = 0$$

prin calibrarea Coulomb $\longrightarrow$ $\nabla\cdot\vec{A} = 0$ şi $\phi = 0$

rămâne $0 = 0$; $$\boxed{\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} = 0} \tag{1.12}$$

sau ecuaţia de câmp EM liber se scrie $$\Box\vec{A} = 0 \tag{1.13}$$

- Soluţiile de undă plană acum sunt: $$\vec{A} = \vec{\epsilon}\, e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} \tag{1.14}$$

- Conform calibrării Coulomb, componentele câmpului sunt normale la direcţia de propagare:

$$\begin{aligned} \nabla\cdot\vec{A} &= \vec{k}\cdot\vec{\epsilon}\; e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} = 0 \\ \text{deci} \quad & \vec{k}\cdot\vec{\epsilon} = 0 \end{aligned} \tag{1.15}$$

- Cu două polarizări lineare transversale independente, ce devin polarizările cuantelor de câmp (fotoni). Acestea se notează cu indicele λ de polarizare: $\vec{\epsilon} \to \vec{\epsilon}_\lambda$

- De obicei, vectorul polarizare este vector unitar, cu componentele normale reciproc: $\vec{\epsilon}_\lambda \cdot \vec{\epsilon}_{\lambda'} = \delta_{\lambda\lambda'}$

- Soluţia generală de câmp electromagnetic este o superpoziţie, o dezvoltare Fourier sau o combinaţie lineară (care se transformă ca un vector 3-dim) de câmpuri de undă plană,

$$\hat{A}_i = \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega} \left(\epsilon_{\lambda_i}\hat{a}_\lambda(\vec{k})\, e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} + \epsilon_{\lambda_i}\hat{a}^\dagger_\lambda(\vec{k})\, e^{i\,(\omega t - \vec{k}\cdot\vec{x})}\right) \tag{1.16}$$

coeficienţii Fourier sunt Hermitic conjugati unul faţa de celălalt, astfel încât câmpul să fie real.

- Câmpul de impulsuri conjugat canonic (1.2) $\hat{\pi}^i = \partial\mathcal{L}/\partial(\partial_0 \hat{A}_i) = \hat{F}_{i0} = \hat{E}^i$ este

$$\begin{aligned}\hat{\pi}^i(x) \;=\; \hat{E}^i &= -\underbrace{\nabla\phi}_{=0} - \frac{\partial}{\partial t}A^i = \\ &= \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\,(-i\,\omega)\Big(\epsilon^i_\lambda \hat{a}_\lambda(\vec{k})\, e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} - \epsilon^i_\lambda \hat{a}^\dagger_\lambda(\vec{k})\, e^{i\,(\omega t - \vec{k}\cdot\vec{x})}\Big)\end{aligned} \tag{1.17}$$

- Pentru a exprima coeficienţii dezvoltării Fourier a câmpului $\hat{A}_i(\vec{x},t)$ (1.16), vom face întâi proiecţia lui pe componentele ortogonale de undă plană:

$$\begin{aligned}\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t - \vec{k}'\cdot\vec{x})}\hat{A}_i(\vec{x},t)} &= \\ &= \sum_\lambda \int d^3\vec{x}\frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\Big[\epsilon_{\lambda i}\, \hat{a}_\lambda(\vec{k})\, e^{i\,(\omega'-\omega)\,t} e^{-i\,(\vec{k}'-\vec{k})\cdot\vec{x}} + \\ &\qquad + \epsilon_{\lambda i}\, \hat{a}^\dagger_\lambda(\vec{k})\, e^{i\,(\omega'+\omega)\,t} e^{-i\,(\vec{k}'+\vec{k})\cdot\vec{x}}\Big]\end{aligned}$$

Integrarea după $d^3\vec{x}$ ne dă,

$$\int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}'-\vec{k})\cdot\vec{x}} = \delta^3(\vec{k}' - \vec{k}) \qquad \text{şi} \qquad \int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}'+\vec{k})\cdot\vec{x}} = \delta^3(\vec{k}' + \vec{k})$$

$$\begin{aligned}\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t - \vec{k}'\cdot\vec{x})}\hat{A}_i(\vec{x},t)} &= \qquad \text{deoarece} \qquad \omega \equiv k_0 = \sqrt{\vec{k}\cdot\vec{k}} \\ &= \sum_\lambda \int\frac{d^3\vec{k}}{2\omega}\Big[\epsilon_{\lambda i}\, \hat{a}_\lambda(\vec{k})\, e^{i\,(\omega'-\omega)\,t}\delta^3(\vec{k}'-\vec{k}) + \\ &\qquad + \epsilon_{\lambda i}\, \hat{a}^\dagger_\lambda(\vec{k})\, e^{i\,(\omega'+\omega)\,t}\delta^3(\vec{k}'+\vec{k})\Big] = \\ &= \sum_\lambda \frac{1}{2\omega'}\Big[\epsilon_{\lambda i}\, \hat{a}_\lambda(\vec{k}') + \epsilon_{\lambda i}\, \hat{a}^\dagger_\lambda(-\vec{k}')\, e^{i\,2\omega' t}\Big]\end{aligned}$$

- Vom face proiecţia acum pe componentele $\vec{\epsilon}_{\lambda'}$ ortogonale ale polarizării:

$$\begin{aligned}\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t - \vec{k}'\cdot\vec{x})}\; \vec{\epsilon}_{\lambda'}\cdot\hat{\vec{A}}(\vec{x},t)} &= \\ &= \sum_\lambda \frac{1}{2\omega'}\Big[\underbrace{\vec{\epsilon}_{\lambda'}\cdot\vec{\epsilon}_\lambda}_{\delta_{\lambda\lambda'}}\, \hat{a}_\lambda(\vec{k}') + \underbrace{\vec{\epsilon}_{\lambda'}\cdot\vec{\epsilon}_\lambda}_{\delta_{\lambda\lambda'}}\, \hat{a}^\dagger_\lambda(-\vec{k}')\, e^{i\,2\omega' t}\Big] \\ &= \sum_\lambda \frac{1}{2\omega'}\Big[\delta_{\lambda\lambda'}\, \hat{a}_\lambda(\vec{k}') + \delta_{\lambda\lambda'}\, \hat{a}^\dagger_\lambda(-\vec{k}')\, e^{i\,2\omega' t}\Big] \\ &= \frac{1}{2\omega'}\Big[\hat{a}_{\lambda'}(\vec{k}') + \hat{a}^\dagger_{\lambda'}(-\vec{k}')\, e^{i\,2\omega' t}\Big]\end{aligned}$$

Odată ce integralele şi sumele au fost efectuate, putem omite semnul prim de la variabile. In final proiecţiile căutate ale câmpului $\hat{\vec{A}}(\vec{x},t)$ sunt:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega t - \vec{k}\cdot\vec{x})}\; \vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t) = \frac{1}{2\omega}\left[\hat{a}_\lambda(\vec{k}) + \hat{a}^\dagger_\lambda(-\vec{k})\, e^{i\,2\omega\, t}\right]} \tag{1.18}$$

- Vom trece să calculăm proiecţia operatorului câmp de impulsuri $\hat{\pi}^i(\vec{x},t)$ întâi pe componentele ortogonale de undă plană:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t-\vec{k}'\cdot\vec{x})}\hat{\pi}^i(\vec{x},t)} =$$

$$= \sum_\lambda \int d^3\vec{x}\frac{d^3\vec{k}}{(2\pi)^3\,2\omega}(-i\omega)\Big[\epsilon^i_\lambda\,\hat{a}_\lambda(\vec{k})\,e^{i\,(\omega'-\omega)\,t}e^{-i\,(\vec{k}'-\vec{k})\cdot\vec{x}} - \epsilon^i_\lambda\,\hat{a}^\dagger_\lambda(\vec{k})\,e^{i\,(\omega'+\omega)\,t}\,e^{-i\,(\vec{k}'+\vec{k})\cdot\vec{x}}\Big]$$

Integrarea după $d^3\vec{x}$ ne dă,

$$\int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}'-\vec{k})\cdot\vec{x}} = \delta^3(\vec{k}'-\vec{k}) \qquad \text{şi} \qquad \int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}'+\vec{k})\cdot\vec{x}} = \delta^3(\vec{k}'+\vec{k})$$

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t-\vec{k}'\cdot\vec{x})}\hat{\pi}^i(\vec{x},t)} = \qquad \text{deoarece} \qquad \omega\equiv k_0=\sqrt{\vec{k}\cdot\vec{k}}$$

$$= \sum_\lambda\int\frac{d^3\vec{k}}{2\omega}(-i\omega)\Big[\epsilon^i_\lambda\,\hat{a}_\lambda(\vec{k})\,e^{i\,(\omega'-\omega)\,t}\delta^3(\vec{k}'-\vec{k}) - \epsilon^i_\lambda\,\hat{a}^\dagger_\lambda(\vec{k})\,e^{i\,(\omega'+\omega)\,t}\delta^3(\vec{k}'+\vec{k})\Big] =$$

$$= \sum_\lambda -\frac{i}{2}\left[\epsilon^i_\lambda\,\hat{a}_\lambda(\vec{k}') - \epsilon^i_\lambda\,\hat{a}^\dagger_\lambda(-\vec{k}')\,e^{i\,2\omega' t}\right]$$

- Vom face proiecţia acum şi pe componentele $\vec{\epsilon}_{\lambda'}$ ortogonale ale polarizării:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t-\vec{k}'\cdot\vec{x})}\;\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{\pi}}(\vec{x},t)} =$$

$$= \sum_\lambda -\frac{i}{2}\Big[\underbrace{\vec{\epsilon}_{\lambda'}\cdot\vec{\epsilon}_\lambda}_{\delta_{\lambda\lambda'}}\,\hat{a}_\lambda(\vec{k}') - \underbrace{\vec{\epsilon}_{\lambda'}\cdot\vec{\epsilon}_\lambda}_{\delta_{\lambda\lambda'}}\,\hat{a}^\dagger_\lambda(-\vec{k}')e^{i\,2\omega' t}\Big]$$

$$= \sum_\lambda -\frac{i}{2}\Big[\delta_{\lambda\lambda'}\,\hat{a}_\lambda(\vec{k}') - \delta_{\lambda\lambda'}\,\hat{a}^\dagger_\lambda(-\vec{k}')\,e^{i\,2\omega' t}\Big]$$

$$= -\frac{i}{2}\Big[\hat{a}_{\lambda'}(\vec{k}') - \hat{a}^\dagger_{\lambda'}(-\vec{k}')\,e^{i\,2\omega' t}\Big]$$

Odată ce integralele şi sumele au fost efectuate, putem omite semnul prim de la variabile. In final proiecţiile căutate ale câmpului $\hat{\vec{\pi}}(\vec{x},t)$ sunt:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega t-\vec{k}\cdot\vec{x})}\;\vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t) = -\frac{i}{2}\left[\hat{a}_\lambda(\vec{k}) - \hat{a}^\dagger_\lambda(-\vec{k})\,e^{i\,2\omega\,t}\right]} \tag{1.19}$$

- In rezumat, cele două integrale cu proiecţiile câmpurilor $\hat{\vec{A}}(\vec{x},t)$ şi $\hat{\vec{\pi}}(\vec{x},t)$ sunt:

$$\begin{aligned}\int d^3\vec{x}\, e^{i(\omega t-\vec{k}\cdot\vec{x})}\ \vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t) &= \frac{1}{2\omega}\Big[\hat{a}_\lambda(\vec{k})\ +\hat{a}^\dagger_\lambda(-\vec{k})\,e^{i\,2\omega\,t}\Big]\\ \int d^3\vec{x}\, e^{i(\omega t-\vec{k}\cdot\vec{x})}\ \vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t) &= -\frac{i}{2}\Big[\hat{a}_\lambda(\vec{k})\ -\hat{a}^\dagger_\lambda(-\vec{k})\,e^{i\,2\omega\,t}\Big]\end{aligned} \tag{1.20}$$

- Folosind cele două relaţii de mai sus (1.20) putem exprima amplitudinile $a_\lambda(\vec{k})$ şi $a^\dagger_\lambda(\vec{k})$ ale dezvoltării Fourier (1.16) a câmpului cuantic electromagnetic $\hat{A}(\vec{x})$,

$$\boxed{\hat{a}_\lambda(\vec{k}) = \int d^3\vec{x}\ e^{i(\omega t-\vec{k}\cdot\vec{x})}\Big[\omega\,\vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t)+i\,\epsilon_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t)\Big]} \tag{1.21}$$

$$\boxed{\hat{a}^\dagger_\lambda(\vec{k}) = \int d^3\vec{x}\ e^{-i(\omega t-\vec{k}\cdot\vec{x})}\Big[\omega\,\vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t)-i\,\vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t)\Big]} \tag{1.22}$$

1.1.2 Relaţii de comutare operatori de câmp electromagnetic

- Să evaluăm relaţiile de comutare $\boxed{\Big[\hat{a}_\lambda(\vec{k}),\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\Big]}$ folosind expresiile amplitudinilor Fourier $\hat{a}_\lambda(\vec{k})$ (1.21) şi $\hat{a}^\dagger_\lambda(\vec{k})$ (1.22), prin operatorii de câmp electromagnetic şi de impuls canonic conjugat la acelaşi timp.

$$\begin{aligned}\boxed{\Big[\hat{a}_\lambda(\vec{k}),\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\Big]} = &\int d^3\vec{x}\ d^3\vec{x}\,'\ e^{i(k\cdot x-k'\cdot x')}\\ &\Big[\overbrace{\Big(\omega\ \underbrace{\vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t)}_{\epsilon^j_\lambda A_j}+i\ \underbrace{\vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t)}_{\epsilon_{\lambda i}\pi^i}\Big)}^{\hat{a}_\lambda(\vec{k})}\overbrace{\Big(\omega\ \underbrace{\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{A}}(\vec{x}\,',t)}_{\epsilon^j_{\lambda'}A_j}-i\ \underbrace{\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{\pi}}(\vec{x}\,',t)}_{\epsilon_{\lambda' i}\pi^i}\Big)}^{\hat{a}^\dagger_\lambda(\vec{k}\,')} -\\ &-\overbrace{\Big(\omega\ \underbrace{\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{A}}(\vec{x}\,',t)}_{\epsilon^j_{\lambda'}A_j}-i\ \underbrace{\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{\pi}}(\vec{x}\,',t)}_{\epsilon_{\lambda' i}\pi^i}\Big)}^{\hat{a}^\dagger_\lambda(\vec{k}\,')}\overbrace{\Big(\omega\ \underbrace{\vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t)}_{\epsilon^j_\lambda A_j}+i\ \underbrace{\vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t)}_{\epsilon_{\lambda i}\pi^i}\Big)}^{\hat{a}_\lambda(\vec{k})}\Big]\end{aligned}$$

unde $k\cdot x = k_\mu x^\mu$ şi $t=t'$.

Acum dacă folosim primele două relaţii de comutare (1.10) (reamintim $\hat{\pi}^i=\hat{E}^i$):

$\Big[\hat{A}_i(\vec{x},t),\hat{A}_j(\vec{x}\,',t)\Big]=0$ şi $\Big[\hat{E}^i(\vec{x},t),\hat{E}^j(\vec{x}\,',t)\Big]=0$

obţinem relaţia de comutare:

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\, e^{i\,(k\cdot x - k'\cdot x')} \times$$

$$\times\, i\,\omega \left(\epsilon_{\lambda i}\, \epsilon^j_{\lambda'} \underbrace{\left[\hat{\pi}^i(\vec{x},t),\; \hat{A}_j(\vec{x}\,',t)\right]} - \epsilon^j_\lambda\, \epsilon_{\lambda' i} \underbrace{\left[\hat{A}_j(\vec{x},t),\; \hat{\pi}^i(\vec{x}\,',t)\right]} \right)$$

Inlocuim comutatorii marcaţi cu acoladă, cu a treia relaţie de comutare (1.10,)

$$\left[\hat{A}_i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] = i \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\delta^j_i - \frac{p_i p^j}{p^2}\right) e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}\,')}$$

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\, \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\,(k\cdot x - k'\cdot x')} \times$$

$$\times\, \omega \left[\epsilon_{\lambda i}\, \epsilon^j_{\lambda'} \left(\delta^i_j - \frac{p_j p^i}{p^2}\right) e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}\,')} + \epsilon^j_\lambda\, \epsilon_{\lambda' i} \left(\delta^i_j - \frac{p_j p^i}{p^2}\right) e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}\,')} \right]$$

Tinând cont de transversalitatea componentelor vectorului polarizare
$\epsilon_{\lambda i}\, \epsilon^j_{\lambda'} = \delta^j_i\, \delta_{\lambda\lambda'}$ şi $\epsilon^j_\lambda\, \epsilon_{\lambda' i} = \delta^i_j\, \delta_{\lambda\lambda'}$

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,' \frac{d^3\vec{p}}{(2\pi)^3} e^{i\,(k\cdot x - k'\cdot x')} \left(2\omega\, \underbrace{\vec{\epsilon}_\lambda \cdot \vec{\epsilon}_{\lambda'}}\; e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}')}\right)$$

Tinând cont de transversalitatea componentelor vectorului polarizare $\vec{\epsilon}_\lambda \cdot \vec{\epsilon}_{\lambda'} = \delta_{\lambda\lambda'}$,

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,' \underbrace{\frac{d^3\vec{p}}{(2\pi)^3}}\, e^{i\,(k\cdot x - k'\cdot x')} \left(2\omega\, \delta_{\lambda\lambda'}\, \underbrace{e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}')}}\right)$$

$$= \int d^3\vec{x}\; d^3\vec{x}\,' e^{i\,(k\cdot x - k'\cdot x')}\, 2\omega\, \delta_{\lambda\lambda'}\, \delta^3(\vec{x} - \vec{x}\,')$$

$$= \int d^3\vec{x}\, e^{i\,(k-k')\cdot x}\, 2\omega\, \delta_{\lambda\lambda'}$$

Final, prin integrarea după $d^3\vec{x}$, comutatorul cautat este:

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right] = (2\pi)^3\, 2\omega\, \delta_{\lambda\lambda'}\, \delta^3(\vec{k} - \vec{k}\,')} \tag{1.23}$$

- In concluzie, setul complet de comutatori pentru operatorii coeficienţi Fourier ai câmpului electromagnetic cuantic, este:

$$\boxed{\begin{aligned} \left[\hat{a}_\lambda(\vec{k}), \hat{a}_{\lambda'}(\vec{k}\,')\right] &= 0 \\ \left[\hat{a}^\dagger_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right] &= 0 \\ \left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right] &= (2\pi)^3\, 2\omega\, \delta_{\lambda\lambda'}\, \delta^3(\vec{k} - \vec{k}\,') \end{aligned}} \tag{1.24}$$