# Procese QED în câmpuri laser intense
# tema 42

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April 8, 2022

.

”Verbum sapienti sat est”

# Cuprins

## 1.1 Cuantificarea câmpului electromagnetic (calibrare Coulomb)

- Cuantificarea câmpului electromagnetic, la fel ca a oricărui câmp clasic, se face prin definirea relaţiilor de comutare între operatorii de câmp, aici $\hat{A}^\mu$, şi cei de impuls canonic conjugat $\hat{\pi}^\mu$.

- Densitatea de Lagrangian a câmpului electromagnetic liber (cu $j_\mu = 0$, scrisă acum cu operatorii de câmp:

$$\mathcal{L} = \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} \tag{1.1}$$

unde operatorul tensor electromagnetic $\hat{F}_{\mu\nu}$, este:

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$$
$$\hat{F}^{\mu\nu} = \partial^\mu \hat{A}^\nu - \partial^\nu \hat{A}^\mu$$

Expresia explicită $\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}$ este:

$$\begin{aligned}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} = {} & 2\left(\partial_1\hat{A}_2 - \partial_2\hat{A}_1\right)^2 + 2\left(\partial_2\hat{A}_3 - \partial_3\hat{A}_2\right)^2 + 2\left(\partial_3\hat{A}_1 - \partial_1\hat{A}_3\right)^2 - \\ & -2\left(\partial_0\hat{A}_1 - \partial_1\hat{A}_0\right)^2 - 2\left(\partial_0\hat{A}_2 - \partial_2\hat{A}_0\right)^2 - 2\left(\partial_0\hat{A}_3 - \partial_3\hat{A}_0\right)^2\end{aligned}$$

- Astfel, câmpul de impulsuri, ca derivata Lagrangian-ului în raport cu derivata temporală a câmpului $\hat{A}_\mu$, este:

$$\hat{\pi}^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0\hat{A}_\mu)} = \frac{1}{4}\left[-4\left(\partial_0\hat{A}_\mu - \partial_\mu\hat{A}_0\right)\right] = -\hat{F}_{0\mu} = \hat{F}_{\mu 0} \tag{1.2}$$

Câmpul de impulsuri se identifică cu componentele electrice ale câmpului electromagnetic:

$$\boxed{\begin{aligned}\hat{\pi}^0 &= 0 \\ \hat{\pi}^i &= \hat{E}^i\end{aligned}} \tag{1.3}$$

- Incercăm la început să facem cuantificarea câmpului electromagnetic căutând relaţiile de comutare între operatorii de câmp $\hat{A}^\mu$ şi cei de câmp de impuls canonic conjugat $\hat{\pi}^\mu$, la fel ca în cazul câmpului scalar real.

- Trecerea de la relaţia de comutare între operatorii din mecanica cuantică $\left[\hat{x}, \hat{p}\right] = i\hbar$ la cea între operatorii de câmpuri cuantice (în unităţi cu $c = \hbar = 1$), ar duce la comutatorii:

$$\left[\hat{A}^\mu(\vec{x}, t), \hat{\pi}^\mu(\vec{y}, t)\right] = i\delta^3(\vec{x} - \vec{y}) \tag{1.4}$$

Dar dacă $\hat{\pi}^0 = 0$, relaţia de comutare între $\hat{A}^0 \equiv \hat{\phi}$ şi valoarea $\hat{\pi}^0 = 0$ nu poate fi diferită de zero. Această înseamnă şi violarea invarianţei la transformarea de calibrare,

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\chi \tag{1.5}$$

Putem alege în acest caz o calibrare particulară, *Calibrarea Coulomb*, care face alegerea câmpului vectorial $\vec{A}$ ca unul cu oscilaţii transversale faţă de direcţia de propagare.

$$\nabla \cdot \vec{A} = 0 \quad \text{şi} \quad A^0 \equiv \phi = 0 \tag{1.6}$$

- Acum putem încerca să scriem ca de obicei, relaţiile de comutare între operatorii de câmp $\hat{A}_i(\vec{x},t)$ şi de impuls canonic $\hat{\pi}^j(\vec{x}\,',t) \equiv \hat{E}^j(\vec{x}\,',t)$, luaţi la acelaşi timp,

$$\left[\hat{A}_i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] = i\,\delta_i^j\,\delta^3(\vec{x}-\vec{x}\,') \tag{1.7}$$

- Să vedem acum ce devine (1.7), după ce câmpul şi coeficienţii dezvoltării Fourier au devenit operatori.

- Dacă păstrăm relaţiile $\nabla\cdot\vec{A}=0$ şi $\nabla\cdot\vec{E}=0$ valabile şi pentru operatori, şi aplicăm operatorul $\nabla$ asupra (1.7), obţinem o contradicţie:

$$0 = \Big[\overbrace{\nabla\cdot\hat{\vec{A}}(\vec{x},t)}^{=0}\,,\ \overbrace{\nabla\cdot\hat{\vec{E}}(\vec{x}\,',t)}^{=0}\Big] = i\,\nabla^2\delta^3(\vec{x}-\vec{x}\,') \neq 0 \tag{1.8}$$

unde $\nabla^2 \equiv \partial_i\partial^i$,

- Pentru rezolvarea acestei inadvertenţe va trebui să exprimăm (1.7) astfel încât membrul drept din (1.8) să se anuleze. Pentru aceasta exprimăm membrul drept din (1.7) sub forma:

$$\left[\hat{A}_i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] = i\,\left(\delta_i^j - \frac{\partial_i\partial^j}{\nabla^2}\right)\delta^3(\vec{x}-\vec{x}\,') \tag{1.9}$$

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Intr-adevăr, aplicând acum operatorul $\nabla$, ca $\partial^i\hat{A}_i$ şi $\partial_j\hat{E}^j$, în membrul drept în paranteză avem:

$$\partial_j\partial^i\left(\delta_i^j - \frac{\partial_i\partial^j}{\nabla^2}\right) = \left(\underbrace{\partial_j\partial^i\delta_i^j}_{\nabla^2} - \partial_j\partial^i\frac{\partial_i\partial^j}{\nabla^2}\right) = \left(\nabla^2 - \frac{\nabla^2\nabla^2}{\nabla^2}\right) = 0$$

Astfel paranteza va fi zero, iar relaţia (1.7), cu $\delta_i^j \to \left(\delta_i^j - \frac{\partial_i\partial^j}{\nabla^2}\right)$, devine compatibilă .

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- Exprimând $\delta^3(\vec{x}-\vec{x}\,') = \int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\,\vec{k}\cdot(\vec{x}-\vec{x}\,')}$ şi efectuând $\partial_i\partial^j$ şi $\nabla$, obţinem:

$$\left[\hat{A}_i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] = i\int\frac{d^3\vec{k}}{(2\pi)^3}\left(\delta_i^j - \frac{k_i k^j}{k^2}\right)e^{i\,\vec{k}\cdot(\vec{x}-\vec{x}\,')} \tag{1.10}$$

- In concluzie, setul complet de relaţii de comutare pentru operatorii de câmp electromagnetic cuantic, luaţi la acelaşi timp, va fi,

$$\boxed{\begin{aligned}\left[\hat{A}_i(\vec{x},t), \hat{A}_j(\vec{x}\,',t)\right] &= 0\\ \left[\hat{E}^i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] &= 0\\ \left[\hat{A}_i(\vec{x},t), \hat{E}_j(\vec{x}\,',t)\right] &= i\int\frac{d^3\vec{k}}{(2\pi)^3}\left(\delta_i^j - \frac{k_i k^j}{k^2}\right)e^{i\,\vec{k}(\vec{x}-\vec{x}\,')}\end{aligned}} \tag{1.11}$$

### 1.1.1 Ecuaţia de câmp electromagnetic în calibrarea Coulomb

- Ecuaţia de câmp electromagnetic liber cu $j^\nu = 0$, în calibrarea Coulomb, este:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0 \\ \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) &= 0 \end{aligned} \tag{1.12}$$

sau $\partial_\mu \left(\partial^\mu A^0 - \partial^0 A^\mu\right) = 0$ ; $\partial_\mu \left(\partial^\mu \vec{A} + \nabla A^\mu\right) = 0$

$$\downarrow \qquad\qquad \downarrow$$

$$\partial_\mu \partial^\mu A^0 - \partial_\mu \partial^0 A^\mu = \quad ; \quad \partial_\mu \partial^\mu \vec{A} + \partial_\mu \nabla A^\mu =$$

$$= \overbrace{\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi}^{} - \overbrace{\left(\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t} \nabla \cdot \vec{A}\right)}^{} = 0 \quad ; \quad = \overbrace{\frac{\partial^2}{\partial t^2} \vec{A} - \nabla^2 \vec{A}}^{} + \overbrace{\frac{\partial}{\partial t} \nabla \phi + \nabla(\nabla \cdot \vec{A})}^{} = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$\underbrace{\nabla^2 \phi}_{=0} + \frac{\partial}{\partial t} \underbrace{\nabla \cdot \vec{A}}_{=0} = 0 \quad ; \quad \frac{\partial^2}{\partial t^2} \vec{A} - \nabla^2 \vec{A} + \frac{\partial}{\partial t} \underbrace{\nabla \phi}_{=0} + \nabla (\underbrace{\nabla \cdot \vec{A}}_{=0}) = 0$$

prin calibrarea Coulomb $\longrightarrow$ $\nabla \cdot \vec{A} = 0$ şi $\phi = 0$

rămâne $0 = 0$ ; 
$$\boxed{\frac{\partial^2}{\partial t^2} \vec{A} - \nabla^2 \vec{A} = 0} \tag{1.13}$$

sau ecuaţia de câmp EM liber se scrie
$$\Box \vec{A} = 0 \tag{1.14}$$

- Soluţiile de undă plană acum sunt:
$$\vec{A} = \vec{\epsilon}\, e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} \tag{1.15}$$

- Conform calibrării Coulomb, componentele câmpului sunt normale la direcţia de propagare:
$$\begin{aligned} \nabla \cdot \vec{A} &= \vec{k} \cdot \vec{\epsilon}\; e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} = 0 \\ \text{deci} \quad & \vec{k} \cdot \vec{\epsilon} = 0 \end{aligned} \tag{1.16}$$

- Cu două polarizări lineare transversale independente, ce devin polarizările cuantelor de câmp (fotoni). Acestea se notează cu indicele $\lambda$ de polarizare:
$$\vec{\epsilon} \longrightarrow \vec{\epsilon}_\lambda$$

- De obicei, vectorul polarizare este vector unitar, cu componentele normale reciproc:
$$\vec{\epsilon}_\lambda \cdot \vec{\epsilon}_{\lambda'} = \delta_{\lambda\lambda'}$$

- Soluţia generală de câmp electromagnetic este o superpoziţie, o dezvoltare Fourier sau o combinaţie lineară (care se transformă ca un vector 3-dim) de câmpuri de undă plană,

$$\hat{A}_i = \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega} \left( \epsilon_{\lambda_i} \hat{a}_\lambda(\vec{k})\, e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} + \epsilon_{\lambda_i} \hat{a}^\dagger_\lambda(\vec{k})\, e^{i\,(\omega t - \vec{k}\cdot\vec{x})} \right) \tag{1.17}$$

coeficienţii Fourier sunt Hermitic conjugati unul faţa de celălalt, astfel încât câmpul să fie real.

- Câmpul de impulsuri conjugat canonic (1.2) $\hat{\pi}^i = \partial\mathcal{L}/\partial(\partial_0 \hat{A}_i) = \hat{F}_{i0} = \hat{E}^i$ este

$$\hat{\pi}^i(x) = \hat{E}^i = -\underbrace{\nabla\phi}_{=0} - \frac{\partial}{\partial t}A^i = \\ = \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\,(-i\,\omega)\Big(\epsilon^i_\lambda \hat{a}_\lambda(\vec{k})\, e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} - \epsilon^i_\lambda \hat{a}^\dagger_\lambda(\vec{k})\, e^{i\,(\omega t - \vec{k}\cdot\vec{x})}\Big) \tag{1.18}$$

- Pentru a exprima coeficienţii dezvoltării Fourier a câmpului $\hat{A}_i(\vec{x},t)$ (1.17), vom face întâi proiecţia lui pe componentele ortogonale $e^{i\,(\omega' t - \vec{k}\,'\cdot\,\vec{x})}$ de undă plană:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t - \vec{k}\,'\cdot\,\vec{x})}\hat{A}_i(\vec{x},t)} = \sum_\lambda \int d^3\vec{x} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\Big[\epsilon_{\lambda i}\,\hat{a}_\lambda(\vec{k})\, e^{i\,(\omega' - \omega)\,t} e^{-i\,(\vec{k}\,' - \vec{k})\cdot\,\vec{x}} + \\ + \epsilon_{\lambda i}\,\hat{a}^\dagger_\lambda(\vec{k})\, e^{i\,(\omega' + \omega)\,t} e^{-i\,(\vec{k}\,' + \vec{k})\cdot\,\vec{x}}\Big]$$

Integrarea după $d^3\vec{x}$ ne dă,

$$\int \frac{d^3\vec{x}}{(2\pi)^3} e^{-i\,(\vec{k}\,' - \vec{k})\cdot\,\vec{x}} = \delta^3(\vec{k}\,' - \vec{k}) \qquad \text{şi} \qquad \int \frac{d^3\vec{x}}{(2\pi)^3} e^{-i\,(\vec{k}\,' + \vec{k})\cdot\,\vec{x}} = \delta^3(\vec{k}\,' + \vec{k})$$

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t - \vec{k}\,'\cdot\,\vec{x})}\hat{A}_i(\vec{x},t)} = \qquad \text{deoarece} \qquad \omega \equiv k_0 = \sqrt{\vec{k}\cdot\vec{k}}$$

$$= \sum_\lambda \int \frac{d^3\vec{k}}{2\omega}\Big[\epsilon_{\lambda i}\,\hat{a}_\lambda(\vec{k})\, e^{i\,(\omega' - \omega)\,t}\delta^3(\vec{k}\,' - \vec{k}) + \\ + \epsilon_{\lambda i}\,\hat{a}^\dagger_\lambda(\vec{k})\, e^{i\,(\omega' + \omega)\,t}\delta^3(\vec{k}\,' + \vec{k})\Big] = \\ = \sum_\lambda \frac{1}{2\omega'}\Big[\epsilon_{\lambda i}\,\hat{a}_\lambda(\vec{k}\,') + \epsilon_{\lambda i}\,\hat{a}^\dagger_\lambda(-\vec{k}\,')\, e^{i\,2\omega' t}\Big]$$

- Vom face proiecţia acum pe componentele $\vec{\epsilon}_{\lambda'}$ ortogonale ale polarizării:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t - \vec{k}\,'\cdot\,\vec{x})}\ \vec{\epsilon}_{\lambda'}\cdot\hat{\vec{A}}(\vec{x},t)} = \\ = \sum_\lambda \frac{1}{2\omega'}\Big[\underbrace{\vec{\epsilon}_{\lambda'}\cdot\vec{\epsilon}_\lambda}_{\delta_{\lambda\lambda'}}\,\hat{a}_\lambda(\vec{k}\,') + \underbrace{\vec{\epsilon}_{\lambda'}\cdot\vec{\epsilon}_\lambda}_{\delta_{\lambda\lambda'}}\,\hat{a}^\dagger_\lambda(-\vec{k}\,')\, e^{i\,2\omega' t}\Big] \\ = \sum_\lambda \frac{1}{2\omega'}\Big[\delta_{\lambda\lambda'}\,\hat{a}_\lambda(\vec{k}\,') + \delta_{\lambda\lambda'}\,\hat{a}^\dagger_\lambda(-\vec{k}\,')\, e^{i\,2\omega' t}\Big] \\ = \frac{1}{2\omega'}\Big[\hat{a}_{\lambda'}(\vec{k}\,') + \hat{a}^\dagger_{\lambda'}(-\vec{k}\,')\, e^{i\,2\omega' t}\Big]$$

Odată ce integralele şi sumele au fost efectuate, putem omite semnul prim de la variabile. In final proiecţiile căutate ale câmpului $\hat{\vec{A}}(\vec{x},t)$ sunt:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega t - \vec{k}\cdot\vec{x})}\ \vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t) = \frac{1}{2\omega}\left[\hat{a}_\lambda(\vec{k}) + \hat{a}^\dagger_\lambda(-\vec{k})\, e^{i\,2\omega\, t}\right]} \tag{1.19}$$

- Vom trece să calculăm şi proiecţia operatorului câmp de impulsuri $\hat{\pi}^i(\vec{x},t)$ întâi pe componentele ortogonale $e^{i\,(\omega' t-\vec{k}'\cdot\vec{x})}$ de undă plană:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t-\vec{k}'\cdot\vec{x})}\hat{\pi}^i(\vec{x},t)} =$$

$$= \sum_\lambda \int d^3\vec{x}\frac{d^3\vec{k}}{(2\pi)^3\,2\omega}(-i\omega)\Big[\epsilon^i_\lambda\,\hat{a}_\lambda(\vec{k})\,e^{i\,(\omega'-\omega)\,t}e^{-i\,(\vec{k}'-\vec{k})\cdot\vec{x}} - \epsilon^i_\lambda\,\hat{a}^\dagger_\lambda(\vec{k})\,e^{i\,(\omega'+\omega)\,t}\,e^{-i\,(\vec{k}'+\vec{k})\cdot\vec{x}}\Big]$$

Integrarea după $d^3\vec{x}$ ne dă,

$$\int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}'-\vec{k})\cdot\vec{x}} = \delta^3(\vec{k}'-\vec{k}) \qquad \text{şi} \qquad \int\frac{d^3\vec{x}}{(2\pi)^3}e^{-i\,(\vec{k}'+\vec{k})\cdot\vec{x}} = \delta^3(\vec{k}'+\vec{k})$$

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t-\vec{k}'\cdot\vec{x})}\hat{\pi}^i(\vec{x},t)} = \qquad \text{deoarece} \qquad \omega\equiv k_0=\sqrt{\vec{k}\cdot\vec{k}}$$

$$= \sum_\lambda\int\frac{d^3\vec{k}}{2\omega}\,(-i\omega)\Big[\epsilon^i_\lambda\,\hat{a}_\lambda(\vec{k})\,e^{i\,(\omega'-\omega)\,t}\delta^3(\vec{k}'-\vec{k}) - \epsilon^i_\lambda\,\hat{a}^\dagger_\lambda(\vec{k})\,e^{i\,(\omega'+\omega)\,t}\delta^3(\vec{k}'+\vec{k})\Big] =$$

$$= \sum_\lambda -\frac{i}{2}\left[\epsilon^i_\lambda\,\hat{a}_\lambda(\vec{k}') - \epsilon^i_\lambda\,\hat{a}^\dagger_\lambda(-\vec{k}')\,e^{i\,2\omega' t}\right]$$

- Vom face proiecţia acum şi pe componentele $\vec{\epsilon}_{\lambda'}$ ortogonale ale polarizării:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega' t-\vec{k}'\cdot\vec{x})}\ \vec{\epsilon}_{\lambda'}\cdot\hat{\vec{\pi}}(\vec{x},t)} =$$

$$= \sum_\lambda -\frac{i}{2}\Big[\underbrace{\vec{\epsilon}_{\lambda'}\cdot\vec{\epsilon}_\lambda}_{\delta_{\lambda\lambda'}}\,\hat{a}_\lambda(\vec{k}') - \underbrace{\vec{\epsilon}_{\lambda'}\cdot\vec{\epsilon}_\lambda}_{\delta_{\lambda\lambda'}}\,\hat{a}^\dagger_\lambda(-\vec{k}')e^{i\,2\omega' t}\Big]$$

$$= \sum_\lambda -\frac{i}{2}\Big[\delta_{\lambda\lambda'}\,\hat{a}_\lambda(\vec{k}') - \delta_{\lambda\lambda'}\,\hat{a}^\dagger_\lambda(-\vec{k}')\,e^{i\,2\omega' t}\Big]$$

$$= -\frac{i}{2}\Big[\hat{a}_{\lambda'}(\vec{k}') - \hat{a}^\dagger_{\lambda'}(-\vec{k}')\,e^{i\,2\omega' t}\Big]$$

Odată ce integralele şi sumele au fost efectuate, putem omite semnul prim de la variabile. In final proiecţiile căutate ale câmpului $\hat{\vec{\pi}}(\vec{x},t)$ sunt:

$$\boxed{\int d^3\vec{x}\, e^{i\,(\omega t-\vec{k}\cdot\vec{x})}\ \vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t) = -\frac{i}{2}\left[\hat{a}_\lambda(\vec{k}) - \hat{a}^\dagger_\lambda(-\vec{k})\,e^{i\,2\omega t}\right]} \tag{1.20}$$

- In rezumat, cele două integrale cu proiecţiile câmpurilor $\hat{\vec{A}}(\vec{x},t)$ şi $\hat{\vec{\pi}}(\vec{x},t)$ sunt:

$$\begin{aligned}\int d^3\vec{x}\, e^{i\,(\omega t-\vec{k}\cdot\vec{x})}\ \vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t) &= \frac{1}{2\omega}\Big[\hat{a}_\lambda(\vec{k})\ +\hat{a}^\dagger_\lambda(-\vec{k})\,e^{i\,2\omega\,t}\Big]\\ \int d^3\vec{x}\, e^{i\,(\omega t-\vec{k}\cdot\vec{x})}\ \vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t) &= -\frac{i}{2}\Big[\hat{a}_\lambda(\vec{k})\ -\hat{a}^\dagger_\lambda(-\vec{k})\,e^{i\,2\omega\,t}\Big]\end{aligned} \tag{1.21}$$

- Folosind cele două relaţii de mai sus (1.21) putem exprima amplitudinile $a_\lambda(\vec{k})$ şi $a^\dagger_\lambda(\vec{k})$ ale dezvoltării Fourier (1.17) a câmpului cuantic electromagnetic $\hat{A}(\vec{x})$,

$$\boxed{\hat{a}_\lambda(\vec{k}) = \int d^3\vec{x}\ e^{i\,(\omega t-\vec{k}\cdot\vec{x})}\Big[\omega\,\vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t)+i\,\epsilon_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t)\Big]} \tag{1.22}$$

$$\boxed{\hat{a}^\dagger_\lambda(\vec{k}) = \int d^3\vec{x}\ e^{-i\,(\omega t-\vec{k}\cdot\vec{x})}\Big[\omega\,\vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t)-i\,\vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t)\Big]} \tag{1.23}$$

### 1.1.2 Relaţii de comutare cu operatori de câmp electromagnetic

- Să evaluăm relaţiile de comutare $\boxed{\Big[\hat{a}_\lambda(\vec{k}),\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\Big]}$ folosind expresiile amplitudinilor Fourier $\hat{a}_\lambda(\vec{k})$ (1.22) şi $\hat{a}^\dagger_\lambda(\vec{k})$ (1.23), prin operatorii de câmp electromagnetic şi de impuls canonic conjugat la acelaşi timp.

$$\begin{aligned}\boxed{\Big[\hat{a}_\lambda(\vec{k}),\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\Big]} =& \int d^3\vec{x}\ d^3\vec{x}\,'\,e^{i\,(k\cdot x-k'\cdot x')}\\ &\Big[\overbrace{\Big(\omega\ \underbrace{\vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t)}_{\epsilon^j_\lambda A_j}+i\ \underbrace{\vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t)}_{\epsilon_{\lambda i}\pi^i}\Big)}^{\hat{a}_\lambda(\vec{k})}\overbrace{\Big(\omega\ \underbrace{\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{A}}(\vec{x}\,',t)}_{\epsilon^j_{\lambda'}A_j}-i\ \underbrace{\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{\pi}}(\vec{x}\,',t)}_{\epsilon_{\lambda' i}\pi^i}\Big)}^{\hat{a}^\dagger_\lambda(\vec{k}\,')}-\\ &-\overbrace{\Big(\omega\ \underbrace{\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{A}}(\vec{x}\,',t)}_{\epsilon^j_{\lambda'}A_j}-i\ \underbrace{\vec{\epsilon}_{\lambda'}\cdot\hat{\vec{\pi}}(\vec{x}\,',t)}_{\epsilon_{\lambda' i}\pi^i}\Big)}^{\hat{a}^\dagger_\lambda(\vec{k}\,')}\overbrace{\Big(\omega\ \underbrace{\vec{\epsilon}_\lambda\cdot\hat{\vec{A}}(\vec{x},t)}_{\epsilon^j_\lambda A_j}+i\ \underbrace{\vec{\epsilon}_\lambda\cdot\hat{\vec{\pi}}(\vec{x},t)}_{\epsilon_{\lambda i}\pi^i}\Big)}^{\hat{a}_\lambda(\vec{k})}\Big]\end{aligned}$$

unde $k\cdot x = k_\mu x^\mu$ şi $t=t'$.

Acum dezvoltăm produsele de câte doi operatori şi folosim primele două relaţii de comutare (1.11) (reamintim $\hat{\pi}^i = \hat{E}^i$):

$$\Big[\hat{A}_i(\vec{x},t),\hat{A}_j(\vec{x}\,',t)\Big]=0 \quad \text{şi} \quad \Big[\hat{E}^i(\vec{x},t),\hat{E}^j(\vec{x}\,',t)\Big]=0$$

obţinem relaţia de comutare:

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\; e^{i\,(k\cdot x - k'\cdot x')} \times$$

$$\times\, i\,\omega \left( \epsilon_{\lambda i}\, \epsilon^j_{\lambda'} \underbrace{\left[\hat{\pi}^i(\vec{x},t),\ \hat{A}_j(\vec{x}\,',t)\right]} - \epsilon^j_\lambda\, \epsilon_{\lambda' i} \underbrace{\left[\hat{A}_j(\vec{x},t),\ \hat{\pi}^i(\vec{x}\,',t)\right]} \right)$$

Inlocuim comutatorii marcaţi cu acoladă, cu a treia relaţie de comutare (1.11,)

$$\left[\hat{A}_i(\vec{x},t), \hat{E}^j(\vec{x}\,',t)\right] = i \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\delta^j_i - \frac{p_i p^j}{p^2}\right) e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}\,')}$$

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,'\; \frac{d^3\vec{p}}{(2\pi)^3}\; e^{i\,(k\cdot x - k'\cdot x')} \times$$

$$\times\, \omega \left[ \epsilon_{\lambda i}\, \epsilon^j_{\lambda'} \left(\delta^i_j - \frac{p_j p^i}{p^2}\right) e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}\,')} + \epsilon^j_\lambda\, \epsilon_{\lambda' i} \left(\delta^i_j - \frac{p_j p^i}{p^2}\right) e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}\,')} \right]$$

Tinând cont de transversalitatea componentelor vectorului polarizare

$\epsilon_{\lambda i}\, \epsilon^j_{\lambda'} = \delta^j_i\, \delta_{\lambda\lambda'}$ şi $\epsilon^j_\lambda\, \epsilon_{\lambda' i} = \delta^i_j\, \delta_{\lambda\lambda'}$

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,' \frac{d^3\vec{p}}{(2\pi)^3} e^{i\,(k\cdot x - k'\cdot x')} \left(2\omega\; \underbrace{\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'}}\; e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}\,')}\right)$$

Tinând cont de transversalitatea componentelor vectorului polarizare $\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'} = \delta_{\lambda\lambda'}$,

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]} = \int d^3\vec{x}\; d^3\vec{x}\,' \underbrace{\frac{d^3\vec{p}}{(2\pi)^3}}\; e^{i\,(k\cdot x - k'\cdot x')} \left(2\omega\, \delta_{\lambda\lambda'}\; \underbrace{e^{i\,\vec{p}\cdot(\vec{x}-\vec{x}\,')}}\right)$$

$$= \int d^3\vec{x}\; d^3\vec{x}\,' e^{i\,(k\cdot x - k'\cdot x')}\, 2\omega\, \delta_{\lambda\lambda'}\, \delta^3(\vec{x}-\vec{x}\,')$$

$$= \int d^3\vec{x}\, e^{i\,(k-k')\cdot x}\, 2\omega\, \delta_{\lambda\lambda'}$$

Final, prin integrarea după $d^3\vec{x}$, comutatorul cautat este:

$$\boxed{\left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right] = (2\pi)^3\, 2\omega\, \delta_{\lambda\lambda'}\, \delta^3(\vec{k}-\vec{k}\,')} \tag{1.24}$$

- In concluzie, setul complet de comutatori pentru operatorii coeficienţi Fourier ai câmpului electromagnetic cuantic, este:

$$\boxed{\begin{aligned} \left[\hat{a}_\lambda(\vec{k}), \hat{a}_{\lambda'}(\vec{k}\,')\right] &= 0 \\ \left[\hat{a}^\dagger_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right] &= 0 \\ \left[\hat{a}_\lambda(\vec{k}), \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right] &= (2\pi)^3\, 2\omega\, \delta_{\lambda\lambda'}\, \delta^3(\vec{k}-\vec{k}\,') \end{aligned}} \tag{1.25}$$

### 1.1.3 Hamiltonian de câmp electromagnetic cuantic

- Cu densitatea de Lagrangian (1.1) $\mathcal{L} = -F_{\mu\nu}F^{\mu\nu}/4$, impulsul de câmp electromagnetic $\hat{\pi}^i = \hat{E}^i$ (1.3), cu legătura $\vec{E} = -\partial\vec{A}/\partial t$, densitatea de Hamiltonian de *câmp electromagnetic* cuantic, este:

$$\mathcal{H}_{em} = \underbrace{\hat{\pi}^i}_{\hat{E}^i}\,\underbrace{\partial_0\hat{A}_i}_{\hat{E}_i} - \mathcal{L} = \hat{E}^i\hat{E}_i + \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} = \\ = \hat{E}^2 + \frac{2}{4}\left(\hat{B}^2 - \hat{E}^2\right) = \frac{1}{2}\left(\hat{E}^2 + \hat{B}^2\right) \tag{1.26}$$

- Hamiltonian-ul de câmp electromagnetic cuantic, este: $$H_{em} = \int d^3\vec{x}\ \mathcal{H}_{em} \tag{1.27}$$

Bazat pe dezvoltarea Fourier a câmpului electromagnetic (1.17), expresiile componentelor electrică $\vec{E} = -\partial\vec{A}/\partial t$ şi magnetică $\vec{B} = \nabla\times\vec{A}$, sunt:

$$\hat{E}^j = -\frac{\partial A^j}{\partial t} = \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}(-i\omega\,\epsilon^j_\lambda)\left(\hat{a}_\lambda(\vec{k})\,e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger_\lambda(\vec{k})\,e^{i\,(\omega t - \vec{k}\cdot\vec{x})}\right) \tag{1.28}$$

$$\hat{B}^j = \left(\vec{\nabla}\times\vec{A}\right)^j = \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega}\left(-\,i(\vec{k}\times\vec{\epsilon}_\lambda)^j\right)\left(\hat{a}_\lambda(\vec{k})\,e^{-i\,(\omega t - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger_\lambda(\vec{k})\,e^{i\,(\omega t - \vec{k}\cdot\vec{x})}\right) \tag{1.29}$$

Inlocuind operatorii $\hat{E}$ şi $\hat{B}$ în densitatea (1.26), Hamiltonianul (1.27) devine:

$$H_{em} = -\frac{1}{2}\sum_{\lambda\lambda'}\int d^3\vec{x}\,\frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\,\frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\Bigg[\omega\omega'\,\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'}\times \\ \times\left(\hat{a}_\lambda(\vec{k})\,e^{-i(\omega t - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger_\lambda(\vec{k})\,e^{i(\omega t - \vec{k}\cdot\vec{x})}\right)\left(\hat{a}_{\lambda'}(\vec{k}\,')\,e^{-i(\omega' t - \vec{k}\,'\cdot\vec{x})} - \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,e^{i(\omega' t - \vec{k}\,'\cdot\vec{x})}\right) + \\ + \left(\vec{k}\times\vec{\epsilon}_\lambda\right)\cdot\left(\vec{k}\,'\times\vec{\epsilon}_{\lambda'}\right)\times \\ \times\left(\hat{a}_\lambda(\vec{k})\,e^{-i(\omega t - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger_\lambda(\vec{k})\,e^{i(\omega t - \vec{k}\cdot\vec{x})}\right)\left(\hat{a}_{\lambda'}(\vec{k}\,')\,e^{-i(\omega' t - \vec{k}\,'\cdot\vec{x})} - \hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,e^{i(\omega' t - \vec{k}\,'\cdot\vec{x})}\right)\Bigg]$$

Efectuăm produsele de operatori,

$$H_{em} = -\frac{1}{2}\sum_{\lambda\lambda'}\int d^3\vec{x}\,\frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\,\frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\Bigg[\omega\omega'\,\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'}\times \\ \times\Big(\hat{a}_\lambda(\vec{k})\hat{a}_{\lambda'}(\vec{k}\,')\ e^{-i(\omega+\omega')t}e^{i(\vec{k}+\vec{k}\,')\cdot\vec{x}} - \hat{a}^\dagger_\lambda(\vec{k})\hat{a}_{\lambda'}(\vec{k}\,')\,e^{i(\omega-\omega')t}e^{-i(\vec{k}-\vec{k}\,')\cdot\vec{x}} - \\ - \hat{a}_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\ e^{-i(\omega-\omega')t}e^{i(\vec{k}-\vec{k}\,')\cdot\vec{x}} + \hat{a}^\dagger_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k}\,')e^{i(\omega+\omega')t}e^{-i(\vec{k}+\vec{k}\,')\cdot\vec{x}}\Big) + \\ + \left(\vec{k}\times\vec{\epsilon}_\lambda\right)\cdot\left(\vec{k}\,'\times\vec{\epsilon}_{\lambda'}\right)\times \\ \times\Big(\hat{a}_\lambda(\vec{k})\hat{a}_{\lambda'}(\vec{k}\,')\,e^{-i(\omega+\omega')t}e^{i(\vec{k}+\vec{k}\,')\cdot\vec{x}} - \hat{a}^\dagger_\lambda(\vec{k})\hat{a}_{\lambda'}(\vec{k}\,')\,e^{i(\omega-\omega')t}e^{-i(\vec{k}-\vec{k}\,')\cdot\vec{x}} - \\ - \hat{a}_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,e^{-i(\omega-\omega')t}e^{i(\vec{k}-\vec{k}\,')\cdot\vec{x}} + \hat{a}^\dagger_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,e^{i(\omega+\omega')t}e^{-i(\vec{k}+\vec{k}\,')\cdot\vec{x}}\Big)\Bigg]$$

Acum să efectuăm integrarea după $d^3\vec{x}$,

$$
\begin{aligned}
H_{em} = -\frac{1}{2}\sum_{\lambda\lambda'}\int \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\Bigg[&\omega\omega'\,\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'}\times\\
\times\Bigg(&\hat{a}_\lambda(\vec{k})\hat{a}_{\lambda'}(\vec{k}\,')\,e^{-i(\omega+\omega')t}(2\pi)^3\delta^3(\vec{k}+\vec{k}\,') - \hat{a}^\dagger_\lambda(\vec{k})\hat{a}_{\lambda'}(\vec{k}\,')\,e^{i(\omega-\omega')t}(2\pi)^3\delta^3(\vec{k}-\vec{k}\,')-\\
&-\hat{a}_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,e^{-i(\omega-\omega')t}(2\pi)^3\delta^3(\vec{k}-\vec{k}\,') + \hat{a}^\dagger_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k}\,')e^{i(\omega+\omega')t}(2\pi)^3\delta^3(\vec{k}+\vec{k}\,')\Bigg)+\\
&+\left(\vec{k}\times\vec{\epsilon}_\lambda\right)\cdot\left(\vec{k}\,'\times\vec{\epsilon}_{\lambda'}\right)\times\\
\times\Bigg(&\hat{a}_\lambda(\vec{k})\hat{a}_{\lambda'}(\vec{k}\,')\,e^{-i(\omega+\omega')t}(2\pi)^3\delta^3(\vec{k}+\vec{k}\,') - \hat{a}^\dagger_\lambda(\vec{k})\hat{a}_{\lambda'}(\vec{k}\,')\,e^{i(\omega-\omega')t}(2\pi)^3\delta^3(\vec{k}-\vec{k}\,')-\\
&-\hat{a}_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,e^{-i(\omega-\omega')t}(2\pi)^3\delta^3(\vec{k}-\vec{k}\,') + \hat{a}^\dagger_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,e^{i(\omega+\omega')t}(2\pi)^3\delta^3(\vec{k}+\vec{k}\,')\Bigg)\Bigg]
\end{aligned}
$$

cu $\omega = \sqrt{\vec{k}\cdot\vec{k}}$ şi proprietăţile funcţiilor $\delta$ din fiecare termen cu exponenţiale, conduc la:

$$
\begin{aligned}
&\delta^3(\vec{k}+\vec{k}\,') \implies \vec{k}\,' = -\vec{k} \quad \text{de unde} \quad \omega' = \omega\\
&\delta^3(\vec{k}-\vec{k}\,') \implies \vec{k}\,' = \vec{k} \quad \text{de unde} \quad \omega' = \omega
\end{aligned}
$$

simplificăm cu $(2\pi)^3$, apoi prin integrare după $d^3\vec{k}\,'$ cu funcţiile $\delta$, ţinând cont că la $\vec{k}\,'$ trebuie să schimbăm semnul pentru termenii cu $\delta^3(\vec{k}+\vec{k}\,')$, obţinem,

$$
\begin{aligned}
H_{em} = -\frac{1}{2}\sum_{\lambda\lambda'}\int \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\frac{1}{2\omega}\Bigg[&\omega^2\,\underbrace{\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'}}_{\delta_{\lambda\lambda'}}\times\\
\times\Bigg(&\hat{a}_\lambda(\vec{k})\,\hat{a}_{\lambda'}(-\vec{k})\,e^{-2i\omega t} - \hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}) - \hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k}) + \hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(-\vec{k})\,e^{2i\omega t}\Bigg)+\\
&+\underbrace{\left(\vec{k}\times\vec{\epsilon}_\lambda\right)\cdot\left(\vec{k}\times\vec{\epsilon}_{\lambda'}\right)}_{\omega^2\delta_{\lambda\lambda'}}\times\\
\times\Bigg(&-\hat{a}_\lambda(\vec{k})\,\hat{a}_{\lambda'}(-\vec{k})\,e^{-2i\omega t} - \hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}) - \hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k}) - \hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(-\vec{k})\,e^{2i\omega t}\Bigg)\Bigg]
\end{aligned}
$$

Facem evaluarea factorului marcat cu acoladă: $\left(\vec{k}\times\vec{\epsilon}_\lambda\right)\cdot\left(\vec{k}\times\vec{\epsilon}_{\lambda'}\right)$. Folosim identitatea:

$$
\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right) = \left(\vec{a}\cdot\vec{c}\right)\left(\vec{b}\cdot\vec{d}\right) - \left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)
$$

folosind şi relaţia de ortogonalitate a componentelor de polarizare $\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'} = \delta_{\lambda\lambda'}$, precum şi condiţia de transversalitate a polarizării $\vec{\epsilon}_\lambda$ pe direcţia de propagare $\vec{k}$, avem,

$$
\left(\vec{k}\times\vec{\epsilon}_\lambda\right)\cdot\left(\vec{k}\times\vec{\epsilon}_{\lambda'}\right) = \left(\vec{k}\cdot\vec{k}\right)\left(\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'}\right) - \left(\vec{k}\cdot\vec{\epsilon}_{\lambda'}\right)\left(\vec{\epsilon}_\lambda\cdot\vec{k}\right) = \vec{k}^{\,2}\,\delta_{\lambda\lambda'} = \omega^2\,\delta_{\lambda\lambda'}
$$

Inlocuind în expresia Hamiltonian-ului, acesta se simplifica semnificativ.

$$H_{em} = -\frac{1}{2}\sum_{\lambda\lambda'}\int \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\frac{1}{2\omega}\,\omega^2\,\delta_{\lambda\lambda'}\,\times$$
$$\times\Bigg(\cancel{\hat{a}_\lambda(\vec{k})\,\hat{a}_{\lambda'}(-\vec{k})}\,e^{-2i\omega t} - \hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}) - \hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k}) + \cancel{\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(-\vec{k})}\,e^{2i\omega t} -$$
$$-\cancel{\hat{a}_\lambda(\vec{k})\,\hat{a}_{\lambda'}(-\vec{k})}\,e^{-2i\omega t} - \hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}) - \hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k}) - \cancel{\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(-\vec{k})}\,e^{2i\omega t}\Bigg)$$

$$H_{em} = -\frac{1}{2}\sum_{\lambda\lambda'}\int \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\frac{1}{2\omega}\,\omega^2\,\delta_{\lambda\lambda'}\,\times$$
$$\times\Big(-\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}) - \hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k}) - \hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}) - \hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k})\Big)$$
$$= \frac{1}{2}\sum_{\lambda\lambda'}\int \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\frac{1}{\omega}\,\omega^2\,\delta_{\lambda\lambda'}\Big(\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}) + \hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k})\Big)$$

Efectuând suma după $\lambda'$, obţinem,

$$H_{em} = \frac{1}{2}\sum_{\lambda}\int \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\,\omega\,\Big(\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_\lambda(\vec{k}) + \underbrace{\hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_\lambda(\vec{k})}\Big) \tag{1.30}$$

Folosind relaţia de comutare (1.25), pentru $\vec{k}=\vec{k}\,'$ şi $\lambda=\lambda'$, aceasta devine

$$\Longrightarrow \qquad \Big[\hat{a}_\lambda(\vec{k})\,,\,\hat{a}^\dagger_\lambda(\vec{k})\Big] = (2\pi)^3\,2\omega$$

acum putem realiza ordonarea normală $\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_\lambda(\vec{k})$ a operatorilor, similar cu cazul câmpului scalar real.

$$H_{em} = \frac{1}{2}\sum_{\lambda}\int \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\,\omega\,\Big(\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_\lambda(\vec{k}) + \overbrace{\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_\lambda(\vec{k}) + (2\pi)^3 2\omega}\Big) \tag{1.31}$$

Deoarece diversele origini (valoarea zero) ale energiei, nu pot fi măsurate, ultimul termen poate fi neglijat. In acest fel, *Hamiltonian-ul normal ordonat de câmp electromagnetic*, exprimat prin operatorii de creare şi anihilare, este:

$$\boxed{H_{em} = \sum_{\lambda}\int \frac{d^3\vec{k}}{(2\pi)^3\,2\omega}\,\omega\,\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_\lambda(\vec{k})} \tag{1.32}$$

**Creare şi anihilare cuante de câmp electromagnetic**

- Să studiem efectul acţiunii operatorilor de amplitudine Fourier $\hat{a}^\dagger_\lambda$ şi $\hat{a}_\lambda$ asupra stărilor proprii de energie.

$$\hat{H}_{em}\,|E_n\rangle = E_n\,|E_n\rangle \tag{1.33}$$

Să trecem la calcul, ţinând cont de relaţiile de comutare (1.25),

$$H_{em}\,\hat{a}_\lambda(\vec{k})\,|E_n\rangle =$$

$$= \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,\hat{a}_{\lambda'}(\vec{k}\,')\,\hat{a}_\lambda(\vec{k})\,|E_n\rangle$$

trecerea $\hat{a}_\lambda(\vec{k})$ peste o poziţie $\rightarrow$ schimbă de semn (1.25),

$$= \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\underbrace{\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,\hat{a}_\lambda(\vec{k})}\,\hat{a}_{\lambda'}(\vec{k}\,')\,|E_n\rangle$$

$$= \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\left(\hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,') - \underbrace{\left[\hat{a}_\lambda(\vec{k}),\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\right]}\right)\hat{a}_{\lambda'}(\vec{k}\,')\,|E_n\rangle$$

folosind relaţia de comutare (1.25),

$$= \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\left(\hat{a}_\lambda(\vec{k})\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,') - \overbrace{(2\pi)^3\,2\omega\,\underbrace{\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}\,')}}\right)\underbrace{\hat{a}_{\lambda'}(\vec{k}\,')}\,|E_n\rangle$$

pentru $\vec{k}\,'=\vec{k} \quad \Rightarrow \quad \omega'=\omega$ sumăm $\lambda'$ şi integrăm $d^3\vec{k}\,'$ ultimul termen

$$= \hat{a}_\lambda(\vec{k})\underbrace{\sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,\hat{a}_{\lambda'}(\vec{k}\,')}_{H_{em}}\,|E_n\rangle - \left(\omega\,\hat{a}_\lambda(\vec{k})\,|E_n\rangle\right)$$

$$= \hat{a}_\lambda(\vec{k})\,\underbrace{H_{em}\,|E_n\rangle}_{=E_n|E_n\rangle} - \omega\,\hat{a}_\lambda(\vec{k})\,|E_n\rangle = \hat{a}_\lambda(\vec{k})\,E_n\,|E_n\rangle - \omega\,\hat{a}_\lambda(\vec{k})\,|E_n\rangle$$

$$= (E_n - \omega)\,\hat{a}_\lambda(\vec{k})\,|E_n\rangle$$

In concluzie,

$$\boxed{H_{em}\,\underbrace{\hat{a}_\lambda(\vec{k})\Big|E_n\Big\rangle}_{|E_{n-1}\rangle} = (E_n-\omega)\,\underbrace{\hat{a}_\lambda(\vec{k})\Big|E_n\Big\rangle}_{|E_{n-1}\rangle}} \tag{1.34}$$

- Deci, operatorul $\hat{a}_\lambda(\vec{k})$ este operator de anihilare sau coborâre pe o stare $\big|E_{n-1}\big\rangle$, cu o cuantă de energie $\hbar\omega$ mai jos, prin acţiunea asupra stării $\big|E_n\big\rangle$,

$$\boxed{\hat{a}_\lambda(\vec{k})\Big|E_n\Big\rangle = \Big|E_{n-1}\Big\rangle} \tag{1.35}$$

- Să trecem şi să căutăm acelaşi lucru în cazul operatorilor $\hat{a}^\dagger_\lambda$ asupra stărilor proprii de energie (1.33).

  Să trecem la calcul, ţinând cont de relaţiile de comutare (1.25),

$$
\begin{aligned}
&H_{em}\,\hat{a}^\dagger_\lambda(\vec{k})\,|E_n\rangle = \\
&\quad = \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,\underbrace{\hat{a}_{\lambda'}(\vec{k}\,')\,\hat{a}^\dagger_\lambda(\vec{k})}\,|E_n\rangle \\
&\quad \text{folosind relaţia de comutare (1.25),} \\
&\quad = \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\left(\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}\,') - \underbrace{\left[\hat{a}^\dagger_\lambda(\vec{k})\,,\,\hat{a}_{\lambda'}(\vec{k}\,')\right]}\right)|E_n\rangle \\
&\quad = \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\left(\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}\,') + \underbrace{\left[\hat{a}_{\lambda'}(\vec{k}\,')\,,\,\hat{a}^\dagger_\lambda(\vec{k})\right]}\right)|E_n\rangle \\
&\quad \text{folosind relaţia de comutare (1.25),} \\
&\quad = \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\left(\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}\,') + \overbrace{(2\pi)^3\,2\omega\,\underbrace{\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}\,')}}\right)|E_n\rangle \\
&\quad \text{pentru} \quad \vec{k}\,'=\vec{k} \quad \Rightarrow \quad \omega'=\omega \quad \text{sumăm } \lambda' \text{ şi integrăm } d^3\vec{k}\,' \text{ ultimul termen} \\
&\quad = \sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\,\hat{a}^\dagger_\lambda(\vec{k})\,\hat{a}_{\lambda'}(\vec{k}\,')\,|E_n\rangle + \left(\omega\,\hat{a}^\dagger_\lambda(\vec{k})\,|E_n\rangle\right) \\
&\quad = \hat{a}^\dagger_\lambda(\vec{k})\sum_{\lambda'}\int \frac{d^3\vec{k}\,'}{(2\pi)^3\,2\omega'}\,\omega'\,\hat{a}^\dagger_{\lambda'}(\vec{k}\,')\hat{a}_{\lambda'}(\vec{k}\,')\,|E_n\rangle + \omega\,\hat{a}^\dagger_\lambda(\vec{k})\,|E_n\rangle\Big) \\
&\quad = \hat{a}^\dagger_\lambda(\vec{k})\,\underbrace{H_{em}\,|E_n\rangle}_{=E_n|E_n\rangle} + \omega\,\hat{a}^\dagger_\lambda(\vec{k})\;|E_n\rangle = \hat{a}^\dagger_\lambda(\vec{k})\,E_n\,|E_n\rangle + \omega\,\hat{a}^\dagger_\lambda(\vec{k})\,|E_n\rangle \\
&\quad = (E_n+\omega)\,\hat{a}^\dagger_\lambda(\vec{k})\,|E_n\rangle
\end{aligned}
$$

  In concluzie,

$$
\boxed{H_{em}\;\underbrace{\hat{a}^\dagger_\lambda(\vec{k})\Big|E_n\Big\rangle}_{|E_{n+1}\rangle} = (E_n+\omega)\;\underbrace{\hat{a}^\dagger_\lambda(\vec{k})\Big|E_n\Big\rangle}_{|E_{n+1}\rangle}} \tag{1.36}
$$

- Deci, operatorul $\hat{a}^\dagger_\lambda(\vec{k})$ este operator de creare a stării $\Big|E_{n+1}\Big\rangle$, sau ridicare pe starea cu o cuantă de energie $\hbar\omega$ mai sus, prin acţiunea asupra stării $\Big|E_n\Big\rangle$,

$$
\boxed{\hat{a}^\dagger_\lambda(\vec{k})\Big|E_n\Big\rangle = \Big|E_{n+1}\Big\rangle} \tag{1.37}
$$